# Redex: Practical Semantics Engineering

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PLT Redex consists of a domain-specific language for specifying reduction semantics, plus a suite of tools for working with the semantics.

This manual consists of four parts: a short tutorial introduction, a long tutorial introduction, a reference manual for Redex, and a description of the Redex automated testing benchmark suite. Also see `http://redex.racket-lang.org/` and the `examples` subdirectory in the `redex` collection.

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# 1 Amb: A Redex Tutorial

This tutorial is designed for those familiar with the call-by-value $\lambda$-calculus (and evaluation contexts), but not Redex. The tutorial works though a model of the $\lambda$-calculus extended with a variation on McCarthy's `amb` operator for ambiguous choice (McCarthy 1963; Zabih et al. 1987).

If you are not familiar with Racket, first try the quick tutorial documentation or the more tutorial documentation. If you wish to follow along with only parts of this tutorial (perhaps using a different IDE or in a text-only context), refer to §4 "The Redex Reference" for the full details on the constructs presented here.

The model includes a standard evaluation reduction relation and a type system. Along the way, the tutorial demonstrates Redex's support for unit testing, random testing, typesetting, metafunctions, reduction relations, and judgment forms. It also includes a number of exercises to use as jumping off points to explore Redex in more depth, and each of the functions and syntactic forms used in the examples are linked to more information.

## 1.1 Defining a Language

To get started, open DrRacket, and put the following two lines at the top of the file (if the first line is not there already, use the Language|Choose Language... menu item to make sure that DrRacket is set to use the language declaration in the source).

```
#lang racket
(require redex)
```

Those lines tell DrRacket that we're writing a program in the `racket` language and we're going to be using the `redex` DSL.

Next, enter the following definition.

```
(define-language L
  (e (e e)
     (λ (x t) e)
     x
     (amb e ...)
     number
     (+ e ...)
     (if0 e e e)
     (fix e))
  (t (→ t t) num)
  (x variable-not-otherwise-mentioned))
```

The `define-language` form gives a name to a grammar. In this case, `L` is the Racket-level name referring to the grammar containing the non-terminal `e`, with eight productions (application, abstraction, variables, `amb` expressions, numbers, addition expressions, if-zero expressions, and fixpoint expressions), the non-terminal `t` with two productions, and the non-terminal `x` that uses the pattern keyword `variable-not-otherwise-mentioned`. This special pattern matches all symbols except those used as literals in the grammar (in this case: $\lambda$, `amb`, `+`, `if0`, `fix`, and $\rightarrow$).

Once we have defined the grammar, we can ask Redex if specific terms match the grammar. This expression checks to see if the `e` non-terminal (from `L`) matches the object-language expression `(λ (x) x)`.

To do this, first click the Run button in DrRacket's toolbar and then enter the following expression after the prompt. For the remainder of this tutorial, expressions prefixed with a > are intended to be run in the interactions window (lower pane), and expressions without the > prefix belong in the definitions window (the upper pane).

```
> (redex-match
   L
   e
   (term (λ (x) x)))
#f
```

In general, a `redex-match` expression first names a language, then a pattern, and then its third position is an arbitrary Racket expression. In this case, we use `term` to construct an Redex object-level expression. The `term` operator is much like Lisp's `quasiquote` (typically written ‘).

This term does not match `e` (since `e` insists the function parameters come with types), so Redex responds with `#f`, false.

When an expression does match, as with this one:

```
> (redex-match
   L
   e
   (term ((λ (x num) (amb x 1))
          (+ 1 2))))
(list
 (match
  (list
   (bind
    'e
    '((λ (x num) (amb x 1)) (+ 1 2))))))
```

Redex responds with bindings for all of the pattern variables. In this case, there is just one, `e`, and it matches the entire expression.

We can also use matching to extract sub-pieces. For example, we can pull out the function and argument position of an application expression like this:

```
> (redex-match
   L
   (e_1 e_2)
   (term ((λ (x num) (amb x 1))
          (+ 1 2))))
(list
 (match
  (list
   (bind 'e_1 '(λ (x num) (amb x 1)))
   (bind 'e_2 '(+ 1 2)))))
```

As you probably noticed, `redex-match` returns a list of matches, not just a single match. The previous matches each only matched a single way, so the corresponding lists only have a single element. But a pattern may be ambiguous, e.g., the following pattern which matches any non-empty sequence of expressions, but binds different elements of the sequence in different ways:

```
> (redex-match
   L
   (e_1 ... e_2 e_3 ...)
   (term ((+ 1 2)
          (+ 3 4)
          (+ 5 6))))
(list
 (match
  (list
   (bind 'e_1 '())
   (bind 'e_2 '(+ 1 2))
   (bind 'e_3 '((+ 3 4) (+ 5 6)))))
 (match
  (list
   (bind 'e_1 '((+ 1 2)))
   (bind 'e_2 '(+ 3 4))
   (bind 'e_3 '((+ 5 6)))))
 (match
  (list
   (bind 'e_1 '((+ 1 2) (+ 3 4)))
   (bind 'e_2 '(+ 5 6))
   (bind 'e_3 '()))))
```

**Exercise 1**

Use `redex-match` to extract the body of the $\lambda$ expression from this object-language program:

```
((λ (x num) (+ x 1))
 17)
```

**Exercise 2**

Use `redex-match` to extract the range portion of the type `(→ num (→ num num))`.

**Exercise 3**

Redex's pattern language supports ambiguity through non-terminals, the `in-hole` pattern, and ellipsis placement (as in the example just above). Use the latter source of ambiguity to design a pattern that matches one way for each adjacent pair of expressions in a sequence. That is, if you match the sequence `(1 2 3 4)`, then you'd expect one match for 1 & 2, one match for 2 & 3, and one match for 3 & 4. In general, this pattern should produce n matches when there are n+1 expressions in the sequence.

To test your solution use `redex-match` like this:

```
(redex-match
 L
  ; your solution goes here
 (term (1 2 3 4)))
```

where you expect a result like this

```
(list
 (match (list (bind 'e_1 1) (bind 'e_2 2)))
 (match (list (bind 'e_1 2) (bind 'e_2 3)))
 (match (list (bind 'e_1 3) (bind 'e_2 4))))
```

but possibly with more pattern variables in the resulting match.

**Exercise 4**

The ellipsis pattern can also be "named" via subscripts that, when duplicated, force the lengths of the corresponding sequences to match. For example, the pattern

```
((λ (x ...) e) v ...)
```

matches application expressions where the function may have a different arity than the number of arguments it receives, but the pattern:

```
((λ (x ..._1) e) v ..._1)
```

ensures that the number of xs is the same as the number of vs.

Use this facility to write a pattern that matches odd length lists of expressions, returning one match for each pair of expressions that are equidistant from the ends of the sequence. For example, if matching the sequence `(1 2 3 4 5)`, there would be two matches, one for the pair `1` & `5` and another for the pair `2` & `4`. Your match should include the bindings `e_left` and `e_right` that extract these pairs (one element of the pair bound to `e_left` and the other to `e_right`). Test your pattern with `redex-match`.

## 1.2 Typing

To support a type system for our language, we need to define type environments, which we do by extending the language `L` with a new non-terminal `Γ`, that we use to represent environments; and by letting the middle dot `·`—not to be confused with a regular dot `.`—represent the empty environment.

```
(define-extended-language L+Γ L
  [Γ · (x : t Γ)])
```

The `define-extended-language` form accepts the name of the new language, the name of the extended language and then a series of non-terminals just like `define-language`.

In the extended language, we can give all of the typing rules for our language. Ignoring the `#:mode` specification for a moment, the beginning of this use of `define-judgment-form` has a contract declaration indicating that the judgments all have the shape `(types Γ e t)`.

```
(define-judgment-form
  L+Γ
  #:mode (types I I O)
  #:contract (types Γ e t)

  [(types Γ e_1 (→ t_2 t_3))
   (types Γ e_2 t_2)
   -------------------------
   (types Γ (e_1 e_2) t_3)]

  [(types (x : t_1 Γ) e t_2)
   ----------------------------------
   (types Γ (λ (x t_1) e) (→ t_1 t_2))]

  [(types Γ e (→ (→ t_1 t_2) (→ t_1 t_2)))
   ---------------------------------------
   (types Γ (fix e) (→ t_1 t_2))]
```

```
 [----------------------
  (types (x : t Γ) x t)]

 [(types Γ x_1 t_1)
  (side-condition (different x_1 x_2))
  -----------------------------------
  (types (x_2 : t_2 Γ) x_1 t_1)]

 [(types Γ e num) ...
  -----------------------
  (types Γ (+ e ...) num)]

 [---------------------
  (types Γ number num)]

 [(types Γ e_1 num)
  (types Γ e_2 t)
  (types Γ e_3 t)
  ------------------------------
  (types Γ (if0 e_1 e_2 e_3) t)]

 [(types Γ e num) ...
  --------------------------
  (types Γ (amb e ...) num)])
```

The first clause gives the typing rule for application expressions, saying that if `e_1` has the type `(→ t_2 t_3)` and `e_2` has the type `t_2`, then the application expression has the type `t_3`.

Similarly, the other clauses give the typing rules for all of the other forms in the language.

Most of the rules use `types`, or give base types to atomic expressions, but the fifth rule is worth a special look. It says that if a variable type checks in some environment, then it also type checks in an extended environment, provided that the environment extension does not use the variable in question.

The `different` function is a metafunction, defined as you might expect:

```
(define-metafunction L+Γ
  [(different x_1 x_1) #f]
  [(different x_1 x_2) #t])
```

The `#:mode` specification tells Redex how to compute derivations. In this case, the mode specification indicates that `Γ` and `e` are to be thought of as inputs, and the type position is to be thought of as an output.

Redex then checks that spec, making sure that, given a particular Γ and e, it can compute a t or, perhaps, multiple ts (if the patterns are ambiguous, or if multiple rules apply to a given pair of Γ and e).

## 1.3 Testing Typing

The judgment-holds form checks to see if a potential judgment is derivable. For example,

```
> (judgment-holds
   (types ·
          ((λ (x num) (amb x 1))
           (+ 1 2))
          t)
   t)
'(num)
```

computes all of the types that the expression

```
((λ (x num) (amb x 1))
 (+ 1 2))
```

has, returning a list of them (in this case, just one).

In general, the judgment-holds form's first argument is an instance of some judgment-form that should have concrete terms for the I positions in the mode spec, and patterns in the positions labeled O. Then, the second position in judgment-holds is an expression that can use the pattern variables inside those O positions. The result of judgment-holds will be a list of terms, one for each way that the pattern variables in the O positions can be filled when evaluating judgment-holds's second position.

For example, if we wanted to extract only the range position of the type of some function, we could write this:

```
> (judgment-holds
   (types ·
          (λ (f (→ num (→ num num))) (f (amb 1 2)))
          (→ t_1 t_2))
   t_2)
'((→ num num))
```

The result of this expression is a singleton list containing the function type that maps numbers to numbers. The reason you see two open parentheses is that Redex exploits Racket's s-expressions to reflect Redex terms as Racket values. Here's another way to write the same value

```
> (list (term (→ num num)))
'((→ num num))
```

Racket's printer does not know that it should use `term` for the inner lists and `list` (or `quote`) for the outer list, so it just uses the `quote` notation for all of them.

We can combine `judgment-holds` with Redex's unit test support to build a small test suite:

```
> (test-equal
   (judgment-holds
    (types · (λ (x num) x) t)
    t)
   (list (term (→ num num))))
> (test-equal
   (judgment-holds
    (types · (amb 1 2 3) t)
    t)
   (list (term num)))
> (test-equal
   (judgment-holds
    (types · (+ 1 2) t)
    t)
   (list (term (→ num num))))
FAILED :26.0
   actual: '(num)
expected: '((→ num num))
```

Redex is silent when tests pass and gives the source location for the failures, as above. The `test-equal` form accepts two expressions, evaluates them, and checks to see if they are `equal?` (structural equality).

To see a summary of the tests run so far, call `test-results`.

```
> (test-results)
1 test failed (out of 3 total).
```

### Exercise 5

Remove the `different` side-condition and demonstrate how one expression now has multiple types, using `judgment-holds`. That is, find a use of `judgment-holds` that returns a list of length two, with two different types in it.

### Exercise 6

The typing rule for `amb` is overly restrictive. In general, it would be better to have a rule like this one:

```
[(types Γ e t) ...
 -----------------------
 (types Γ (amb e ...) t)]
```

but Redex does not support this rule because the mode specification is not satisfied in the case that `amb` has no subexpressions. That is, any type should be okay in this case, but Redex cannot "guess" which type is the one needed for a particular derivation, so it rejects the entire `define-judgment-form` definition. (The error message is different, but this is the ultimate cause of the problem.)

Fix this by annotating `amb` expressions with their types, making suitable changes to the language as well as the `define-judgment-form` for `types`. Add new test cases to make sure you've done this properly.

## 1.4 Defining a Reduction Relation

To reduce terms, Redex provides `reduction-relation`, a form that defines unary relations by cases. To define a reduction relation for our `amb` language, we first need to define the evaluation contexts and values, so we extend the language a second time.

```
(define-extended-language Ev L+Γ
  (p (e ...))
  (P (e ... E e ...))
  (E (v E)
     (E e)
     (+ v ... E e ...)
     (if0 E e e)
     (fix E)
     hole)
  (v (λ (x t) e)
     (fix v)
     number))
```

To give a suitable notion of evaluation for `amb`, we define `p`, a non-terminal for programs. Each program consists of a sequence of expressions and we will use them to represent the possible ways in which an `amb` expression could have been evaluated. Initially, we will simply wrap an expression in a pair of parentheses to generate a program that consists of that single expression.

The non-terminal `P` gives the corresponding evaluation contexts for `p`s and says that evaluation can occur in any of them, without restriction. The grammar for `E` dictates that reduction may occur inside application expressions and addition expressions, always from left to right.

To prepare for the reduction relation, we first define a metafunction for summation.

```
(define-metafunction Ev
  Σ : number ... -> number
  [(Σ number ...)
   ,(apply + (term (number ...)))])
```

This lifts the Racket function `+` to Redex, giving it the name `Σ`. The unquote (comma) in the definition of the metafunction escapes into Racket, using `apply` and `+` to sum up the sequence of numbers that were passed to `Σ`. As we've noted before, the `term` operator is like Racket's `quasiquote` operator, but it is also sensitive to Redex pattern variables. In this case, `(term (number ...))` produces a list of numbers, extracting the arguments from the call to `Σ`.

To define a reduction relation, we also have to define substitution. Generally speaking, substitution functions are tricky to get right and, since they generally are not shown in papers, we have defined a workhorse substitution function in Racket that runs in near linear time. The source code is included with Redex. If you'd like to have a look, evaluate the expression below in the REPL to find the precise path on your system:

```
(collection-file-path "tut-subst.rkt" "redex")
```

(Test cases are in `"test/tut-subst-test.rkt"`, relative to `"tut-subst.rkt"`.)

That file contains the definition of the function `subst/proc`, which expects four arguments: a predicate for determining if an expression is a variable, a list of variables to replace, a list of terms to replace them with, and a term to do the replacement inside (the function has a hard-wired notion of the shape of all binding forms, but is agnostic to the other expression forms in the language).

To use this substitution function, we also need to lift it into Redex, just like we did for `Σ`.

```
(require redex/tut-subst)
(define-metafunction Ev
  subst : x v e -> e
  [(subst x v e)
   ,(subst/proc x? (list (term x)) (list (term v)) (term e))])
(define x? (redex-match Ev x))
```

In this case, we use `term` to extract the values of the Redex variables `x`, `v`, and `e` and then pass them to `subst/proc`.

The definition of `x?` uses a specialized, more efficient form of `redex-match`; supplying `redex-match` with only two arguments permits Redex to do some processing of the pattern, and it results in a predicate that matches the pattern in the given language (which we can supply directly to `subst/proc`).

Using that substitution function, we can now give the reduction relation.

```
(define red
  (reduction-relation
   Ev
   #:domain p
   (--> (in-hole P (if0 0 e_1 e_2))
        (in-hole P e_1)
        "if0t")
   (--> (in-hole P (if0 v e_1 e_2))
        (in-hole P e_2)
        (side-condition (not (equal? 0 (term v))))
        "if0f")
   (--> (in-hole P ((fix (λ (x t) e)) v))
        (in-hole P (((λ (x t) e) (fix (λ (x t) e))) v))
        "fix")
   (--> (in-hole P ((λ (x t) e) v))
        (in-hole P (subst x v e))
        "βv")
   (--> (in-hole P (+ number ...))
        (in-hole P (Σ number ...))
        "+")
   (--> (e_1 ... (in-hole E (amb e_2 ...)) e_3 ...)
        (e_1 ... (in-hole E e_2) ... e_3 ...)
        "amb")))
```

The `reduction-relation` form accepts the name of a language, the domain of the relation (`p` in this case), and then a series of rewriting rules, each of the form `(--> pattern pattern)`.

The first rule replaces `if0` expressions when the test position is `0` by the second subexpression (the true branch). It uses the `in-hole` pattern, the Redex notation for context decomposition. In this case, it decomposes a program into some `P` with an appropriate `if0` expression inside, and then the right-hand side of the rule places `e_1` into the same context.

The rule for the false branch should apply when the test position is any value except `0`. To establish this, we use a `side-condition`. In general, a side-condition is a Racket expression that is evaluated for expressions where the pattern matches; if it returns true, then the rule fires. In this case, we use `term` to extract the value of `v` and then compare it with `0`.

To explore the behavior of a reduction relation, Redex provides `traces` and `stepper`. They both accept a reduction relation and a term, and then show you how that term reduces in a GUI. The GUI that `traces` uses is better suited to a quick overview of the reduction graph and `stepper` is better for more detailed explorations of reduction graphs that have larger expressions in them.

**Exercise 7**

Evaluate

```
(traces red
        (term ((+ (amb 1 2)
                  (amb 10 20)))))
```

It does not show all of the terms by default, but one click the Reduce button shows them all.

If you have Graphviz installed, Redex can use it to lay out the graph; click Fix Layout and Redex will call out to `dot` to lay out the graph.

**Exercise 8**

Design a function that accepts a number `n` and evaluates (ambiguously) to any of the numbers between `n` and `0`. Call it with `10` and look at the results in both `traces` and `stepper`.

Hint: to subtract 1 from `n`, use `(+ n -1)`

## 1.5 Testing Reduction Relations

Redex provides `test-->>` for using testing the transitive closure of a reduction relation. If you supply it a reduction relation and two terms, it will reduce the first term and make sure that it yields the second.

```
> (test-->>
   red
   (term ((if0 1 2 3)))
   (term (3)))
> (test-->>
   red
   (term ((+ (amb 1 2)
             (amb 10 20))))
   (term (11 21 12 22)))
> (test-results)
Both tests passed.
```

The `test-->` form is like `test-->>`, except that it only reduces the term a single step.

```
> (test-->
   red
   (term ((+ (amb 1 2) 3)))
   (term ((+ 1 3) (+ 2 3))))
> (test-results)
One test passed.
```

If a term produces multiple results, then each of the results must be listed.

```
> (test-->
   red
   (term ((+ 1 2) (+ 3 4)))
   (term (3 (+ 3 4)))
   (term ((+ 1 2) 7)))
> (test-results)
One test passed.
```

Technically, when using `test-->>`, it finds all irreducible terms that are reachable from the given term, and expects them all to be listed, with one special case: when it detects a cycle in the reduction graph, then it signals an error. (Watch out: when the reduction graph is infinite and there are no cycles, then `test-->>` consumes all available memory.)

```
> (test-->>
   red
   (term (((fix (λ (x (→ num num)) x)) 1))))
FAILED :45.0
found a cycle in the reduction graph
> (test-results)
1 test failed (out of 1 total).
```

To suppress this behavior, pass `#:cycles-ok` to `test-->>`.

```
> (test-->>
   red #:cycles-ok
   (term (((fix (λ (x (→ num num)) x)) 1))))
> (test-results)
One test passed.
```

This test case has no expected results but still passes, since there are no irreducible terms reachable from the given term.

**Exercise 9**

Extend $\lambda$ to support multiple arguments. Use the notation `(λ (x t) ... e)` for multi-arity $\lambda$ expressions because the `subst/proc` function works properly with $\lambda$ expressions of that shape. Use this definition of `subst`.

```
(define-metafunction Ev
  subst : (x v) ... e -> e
  [(subst (x v) ... e)
   ,(subst/proc x?
                (term (x ...))
                (term (v ...))
                (term e))])
```

Also, adjust the typing rules (and do not forget that an ellipsis can be named, as discussed in exercise 4).

## 1.6 Random Testing

Random testing is a cheap and easy way to find counter-examples to false claims. Unsurprisingly, it is hard to pin down exactly which false claims that random testing can provide counter-examples to. Hanford (1970) put it best (calling his random test case generator a syntax machine): "[a]lthough as a writer of test cases, the syntax machine is certainly unintelligent, it is also uninhibited. It can test a [language] processor with many combinations that would not be thought of by a human test case writer."

To get a sense of how random testing works, we define this Racket predicate

```
(define (progress-holds? e)
  (if (types? e)
      (or (v? e)
          (reduces? e))
      #t))
```

that captures the statement of the progress result.

The three helper functions `types?`, `v?`, and `reduces?` can be defined by using our earlier definitions of typing, the grammar, and the reduction relation, plus calls into Redex:

```
(define (types? e)
  (not (null? (judgment-holds (types · ,e t)
                              t))))

(define v? (redex-match Ev v))

(define (reduces? e)
  (not (null? (apply-reduction-relation
               red
               (term (,e))))))
```

The only new construct here is `apply-reduction-relation`, which accepts a reduction and a term, and returns a list of expressions that it reduces to in a single step. Thus, `reduces?` returns `#t` when the given term is reducible and `#f` otherwise.

Putting all of that together with `redex-check` will cause Redex to randomly generate 1,000 `e`s and attempt to falsify them:

```
> (redex-check Ev e (progress-holds? (term e)))
redex-check: no counterexamples in 1000 attempts
```

The `redex-check` form accepts the name of a language (`Ev` in this case), a pattern (`e` in this case), and a Racket expression that returns a boolean. It randomly generates expressions matching the pattern and then invokes the expression in an attempt to elicit `#f` from the Racket expression.

We can also ask `redex-check` how good of a job it is doing. Specifically, this expression re-runs the same random test, but this time sets up some instrumenting infrastructure to determine how many of the reduction rules fire during the testing. In this case, we create a coverage value that indicates that we're interested in how many of the rules in `red` fired, and then we install it using the `relation-coverage` parameter. In the dynamic extent of the `parameterize`, then, the relation will record how it gets tested. Once that returns we can use `covered-cases` to see exactly how many times each case fired.

```
> (let ([c (make-coverage red)])
    (parameterize ([relation-coverage (list c)])
      (redex-check Ev e (progress-holds? (term e))))
    (covered-cases c))
redex-check: no counterexamples in 1000 attempts
'(("+" . 30)
  ("amb" . 31)
  ("fix" . 0)
  ("if0f" . 1)
  ("if0t" . 1)
  ("βv" . 2))
```

Not many of them! To improve coverage, we can tell `redex-check` to try generating expressions using the patterns on the left-hand side of the rules to generate programs, and then check to see if progress for each of the expressions in the program:

```
> (check-reduction-relation
   red
   (λ (p) (andmap progress-holds? p)))
check-reduction-relation: no counterexamples in 6000 attempts
(tried 1000 attempts with each clause)
```

The `check-reduction-relation` is a shorthand for using `redex-check` to generate elements of the domain of the given reduction relation (`red` in this case), and then pass them to the given function, attempting to elicit `#f`.

In this case, since the domain of `red` is `p`, the random generator produces sequences of `e` expressions, which are reflected into Redex as lists, and so we simply try to see if progress holds for each element of the list, using `andmap`.

Still no test failures, but installing the same coverage testing boilerplate around the call to `check-reduction-relation` tells us that we got much better coverage of the reduction system.

```
> (let ([c (make-coverage red)])
    (parameterize ([relation-coverage (list c)])
      (check-reduction-relation
       red
       (λ (p) (andmap progress-holds? p)))
      (covered-cases c)))
check-reduction-relation: no counterexamples in 6000 attempts
(tried 1000 attempts with each clause)
'(("+" . 2055)
  ("amb" . 2157)
  ("fix" . 1)
  ("if0f" . 272)
  ("if0t" . 104)
  ("βv" . 50))
```

**Exercise 10**

Remove one of the productions from `E` (except `hole`) and find an expression in the revised system that causes `progress?` to return `#f`.

See if `redex-check` can also falsify progress for the same system.

**Exercise 11**

Formulate and randomly check type preservation. Usually, this lemma says that if an expression has a type and it takes a step, then it produces an expression with the same type. In this case, however, formulate a predicate that accepts an expression and checks that, if it has a type and takes a step, then all of the resulting expressions in the new program have the same type.

## 1.7 Typesetting the Reduction Relation

Redex's typesetting facilities accept languages, metafunctions, reduction relations, and judgment-forms and produce typeset output that can be included directly into a figure in a paper.

```
> (render-reduction-relation red)
```

$P[(\mathsf{if0}\ 0\ e_1\ e_2)] \longrightarrow P[e_1]$ [if0t]

$P[(\mathsf{if0}\ v\ e_1\ e_2)] \longrightarrow P[e_2]$ [if0f]
where `(not (equal? 0 v))`

$P[((\mathsf{fix}\ (\lambda\ (x\ t)\ e))\ v)] \longrightarrow P[(((\lambda\ (x\ t)\ e)\ (\mathsf{fix}\ (\lambda\ (x\ t)\ e)))\ v)]$ [fix]

$P[((\lambda\ (x\ t)\ e)\ v)] \longrightarrow P[\mathsf{subst}[\![x, v, e]\!]]$ [βv]

$P[(+\ number\ \ldots)] \longrightarrow P[\Sigma[\![number, \ldots]\!]]$ [+]

$(e_1\ \ldots\ E[(\mathsf{amb}\ e_2\ \ldots)]\ e_3\ \ldots) \longrightarrow (e_1\ \ldots\ E[e_2]\ \ldots\ e_3\ \ldots)$ [amb]

The result of `render-reduction-relation` is rendered directly in DrRacket's interactions window, and also can be saved as a `".ps"` file by passing the name of the file as the second argument to `render-reduction-relation`.

Redex's typesetting also interoperates with the `pict` library. If we pull it in with a `require`:

```
(require pict)
```

then we can use the pict primitives to combine typeset fragments into a larger whole.

```
> (scale (vl-append
          20
          (language->pict Ev)
          (reduction-relation->pict red))
         3/2)
```

$p$ ::= $(e \ \ldots)$
$P$ ::= $(e \ \ldots \ E \ e \ \ldots)$
$E$ ::= $(v \ E)$
| $(E \ e)$
| $(+ \ v \ \ldots \ E \ e \ \ldots)$
| $(\mathsf{if0} \ E \ e \ e)$
| $(\mathsf{fix} \ E)$
| $[]$
$v$ ::= $(\lambda \ (x \ t) \ e)$
| $(\mathsf{fix} \ v)$
| $number$

$P[(\mathsf{if0} \ 0 \ e_1 \ e_2)] \longrightarrow$ [if0t]
$P[e_1]$

$P[(\mathsf{if0} \ v \ e_1 \ e_2)] \longrightarrow$ [if0f]
$P[e_2]$
where `(not (equal? 0 v))`

$P[((\mathsf{fix} \ (\lambda \ (x \ t) \ e)) \ v)] \longrightarrow$ [fix]
$P[(((\lambda \ (x \ t) \ e) \ (\mathsf{fix} \ (\lambda \ (x \ t) \ e))) \ v)]$

$P[((\lambda \ (x \ t) \ e) \ v)] \longrightarrow$ [βv]
$P[\mathsf{subst}[\![x, \ v, \ e]\!]]$

$P[(+ \ number \ \ldots)] \longrightarrow$ [+]
$P[\Sigma[\![number, \ \ldots]\!]]$

$(e_1 \ \ldots \ E[(\mathsf{amb} \ e_2 \ \ldots)] \ e_3 \ \ldots) \longrightarrow$ [amb]
$(e_1 \ \ldots \ E[e_2] \ \ldots \ e_3 \ \ldots)$

Generally speaking, Redex has reasonable default ways to typeset its definitions, except when they escape to Racket. In that case, it typesets the code in a fixed-width font and makes the background pink to call attention to it. While it is possible to use `with-unquote-rewriter` to tell Redex how to typeset those regions, often it is easier to define a metafunc-

tion and call it. In this case, we can use `different` (defined earlier).

```
(define if0-false-rule
  (reduction-relation
   Ev #:domain p
   (--> (in-hole P (if0 v e_1 e_2))
        (in-hole P e_2)
        (side-condition (term (different v 0)))
        "if0f")))
```

Now when we typeset this reduction-relation there is no pink.

```
> (render-reduction-relation if0-false-rule)
```

$P[(\mathsf{if0}\ v\ e_1\ e_2)] \longrightarrow [\mathsf{if0f}]$
$P[e_2]$
where $\mathsf{different}[\![v, 0]\!]$

Still, the typesetting is non-optimal, so we can use `with-compound-rewriter` to adjust the way calls to `different` typeset.

```
> (with-compound-rewriter
    'different
    (λ (lws)
      (list "" (list-ref lws 2) " ≠ " (list-ref lws 3) ""))
    (render-reduction-relation if0-false-rule))
```

$P[(\mathsf{if0}\ v\ e_1\ e_2)] \longrightarrow [\mathsf{if0f}]$
$P[e_2]$
where $v \neq 0$

The compound rewriter is given a list of `lw` structs that correspond to the untypeset sequence for a use of `different`, and then can replace them with a different set of strings and `lw`s. For more details on the structure of `lw` structs and to experiment with them, see `to-lw`.

**Exercise 12**

Redex uses the indentation and newlines in the program source code to determine where the line breaks in the printed output goes, instead of using a pretty-printer, so as to give Redex programmers fine-grained control over how their models typeset.

Exploit this facility so that this expression produces an expression with a minimum amount of whitespace within its bounding box. (The call to `frame` helps to clarify where the bounding box is.)

```
(frame
 (vl-append
```

```
20
(language->pict Ev)
(reduction-relation->pict red)))
```

That is, adjust the whitespace in `Ev` so that it fills as much of the width established by rendering `red`.

**Exercise 13**

Use `render-judgment-form` to typeset `types`. Use a compound rewriter so a use of `(type Γ e t)` is rendered as

$$\Gamma \vdash e : t$$

# 2 Long Tutorial

This tutorial is derived from a week-long Redex summer school, run July 27–31, 2015 at the University of Utah.

## 2.1 The Theoretical Framework

**Goals**
— abstract syntax
— notions of reduction, substitution
— reductions and calculations
— semantics
— standard reduction
— abstract register machines
— types

The lambda calculus:

```
e = x | (\x.e) | (e e)
```

Terms vs trees, abstract over concrete syntax

Encode some forms of primitives: numbers, booleans – good for theory of computation; mostly irrelevant for PL. extensions with primitive data

```
e = x | (\x.e) | (e e) | tt | ff | (if e e e)
```

What we want: develop LC as a *model* of a PL. Because of history, this means two things: a simple logic for calculating with the terms of the language `e == e'` and a system for determining the value of a program. The former is the *calculus*, the latter is the *semantics*.

Both start with basic notions of reduction (axioms). They are just relation on terms:

```
((\x.e) e') beta e[x=e']
```

pronounced: e with x replaced by e'

```
((\x.e) e') beta [e'/x]e
```

pronounced substitute e' for x in e

**Think** substitution via tree surgery, preserving bindings

Here are two more, often done via external interpretation functions ($\delta$)

```
(if tt e e') if-tt e


(if ff e e') if-ff e'
```

If this is supposed to be a theory of functions (and if expressions) we need to be able to use this relations *in context*

```
    e xyz e'
--------------
    e  =  e'

    e = e'                e = e'                e = e'
--------------        --------------        --------------
e e'' = e' e''        e'' e = e'' e'         \x.e  = \x.e'
```

plus reflexivity, symmetry, and transitivity

for any relation `xyz`

Now you have an equational system. what's it good for? you can prove such facts as

```
e (Y e) = (Y e)
```

meaning *every single term has a fixpoint*

All of the above is mathematics but it is just that, mathematics. It might be considered theory of computation, but it is not theory of programming languages. But we can use these ideas to create a theory of programming languages. Plotkin's 1974 TCS paper on call-by-name versus call-by-value shows how to create a theory of programming languages.

In addition, Plotkin's paper also sketches several research programs, mostly on scaling up his ideas to the full spectrum of languages but also on the precise connection between by-value and by-name their relationship, both at the proof-theoretical level as well as at the model-theoretic level.

Here is Plotkin's idea as a quasi-algorithm:

1. Start from an abstract syntax, plus notions of scope and scope-preserving substitution. Consider closed terms *Program*s.

2. Identify a subset of expressions as *Value*s. Use `v` to range over Values.

   **Note** The complement of this set was (later) dubbed *computations*, due to Moggi's work under Plotkin.

3. Define basic notions of reduction (axioms). Examples:

   ```
   ((\x.e) e') beta-name e[x=e']
   ```

   ```
   ((\x.e) v) beta-value e[x=v]
   ```

4. Inductively generate an equational theory from the basic notions of reduction.

5. This theory defines a semantics, that is, a relation *eval* from programs to values:

   ```
   eval : Program x Value
   ```

   **def** `e eval v iff e = v`

6. Prove that *eval* is a function, and you have got yourself a *specification* of an interpreter.

   ```
   eval : Program -> Value
   ```

   ```
   eval(e) = v
   ```

   **Note** This step often reuses a variant of the Church-Rosser theorem of the mathematical theory of lambda calculus.

7. Prove that the calculus satisfies a Curry-Feys standard reduction property. This gives you a second semantics:

   ```
   eval-standard : Program -> Value
   ```

   **def** `eval-standard(e) = v iff e standard reduces to v`

   The new semantics is correct:

   **Theorem** `eval-standard = eval`

   *Standard reduction* is a strategy for the lambda calculus, that is, a function that picks the next reducible expression (called *redex*) to reduce. Plotkin specifically uses the leftmost-outermost strategy but others may work, too.

Plotkin also shows—on an ad hoc basis—that this evaluator function is equivalent to Landin's evaluator based on the SECD machine, an abstract register machine.

Plotkin (following Morris, 1968) uses step 6 from above to add two ideas:

- The interpreter of a programming language (non-constructively) generates a theory of equivalence on phrases.

  **def** `e ~ e' iff placing e and e' into any context yields programs that produce the same observable behavior according to eval`

  **Theorem** ~ is the coarsest equivalence theory and thus unique.

  Let's call ~ the *Truth*.

- **Theorem** e = e' implies e ~ e'. Naturally the reverse doesn't hold.

Matthias's (post)dissertation research extends Plotkin's work in two directions:

1. Plotkin's "algorithm" applies to imperative programming language, especially those extending the lambda calculus syntax with (variable) assignment and non-local control operators.

   §2.8 "Imperative Extensions" explains how two of these work.

2. It is possible to derive useful abstract register machines from the standard reduction semantics of the programming language

   Each machine `M` defines a new semantics:

   **def** `eval-M(e) = v iff load M with e, run, unload, yields v`

   For each of these functions, we can prove an equivalence theorem.

   **Theorem** `eval-M = eval-standard = eval`

His work also shows how this approach greatly simplifies proofs of consistency for the semantics of programming languages and especially so-called type soundness theorems.

## 2.2 Syntax and Metafunctions

**Goals**
— Redex versus Racket
— define languages
— develop metafunctions, includes basic testing, submodules
— extend languages
— generalizing with `any`

### 2.2.1 Developing a Language

To start a program with Redex, start your file with

```
#lang racket
(require redex)
```

The `define-language` from specifies syntax trees via tree grammars:

```
(define-language Lambda
  (e ::= x
         (lambda (x ...) e)
         (e e ...))
  (x ::= variable-not-otherwise-mentioned))
```

The trees are somewhat concrete, which makes it easy to work with them, but it is confusing on those incredibly rare occasions when we want truly abstract syntax.

We can include literal numbers (all of Racket's numbers, including complex) or integers (all of Racket's integers) or naturals (all of Racket's natural numbers)—and many other things.

After you have a syntax, use the grammar to generate instances and check them (typos do sneak in). Instances are generated with `term`:

```
> (define e1 (term y))
> (define e2 (term (lambda (y) y)))
> (define e3 (term (lambda (x y) y)))
> (define e4 (term (,e2 ,e3)))
> e4
'((lambda (y) y) (lambda (x y) y))
```

Mouse over `define`. It is *not* a Redex form, it comes from Racket. Take a close look at the last definition. Comma anyone?

```
(redex-match? Lambda e e4)
```

Define yourself a predicate that tests membership:

```
(define lambda? (redex-match? Lambda e))
```

Now you can formulate language tests:

```
(test-equal (lambda? e1) #true)
(test-equal (lambda? e2) #true)
```

```
(test-equal (lambda? e3) #true)
(test-equal (lambda? e4) #true)

(define eb1 (term (lambda (x x) y)))
(define eb2 (term (lambda (x y) 3)))

(test-equal (lambda? eb1) #false)
(test-equal (lambda? eb2) #false)

(test-results)
```

Make sure your language contains the terms that you want and does *not* contain those you want to exclude. Why should `eb1` and `eb2` not be in `Lambda`'s set of expressions?

### 2.2.2 Developing Metafunctions

To make basic statements about (parts of) your language, define metafunctions. Roughly, a metafunction is a function on the terms of a specific language.

We don't want parameter sequences with repeated variables. Can we say this with a metafunction?

```
(define-metafunction Lambda
  unique-vars : x ... -> boolean)
```

The second line is a Redex contract, not a type. It says `unique-vars` consumes a sequence of `x`s and produces a `boolean`.

How do we say we don't want repeated variables? With patterns.

```
(define-metafunction Lambda
  unique-vars : x ... -> boolean
  [(unique-vars) #true]
  [(unique-vars x x_1 ... x x_2 ...) #false]
  [(unique-vars x x_1 ...) (unique-vars x_1 ...)])
```

Patterns are powerful. More later.

But, don't just define metafunctions, develop them properly: state what they are about, work through examples, write down the latter as tests, *then* define the function.

```
; are the identifiers in the given sequence unique?

(module+ test
```

```
  (test-equal (term (unique-vars x y)) #true)
  (test-equal (term (unique-vars x y x)) #false))

(define-metafunction Lambda
  unique-vars : x ... -> boolean
  [(unique-vars) #true]
  [(unique-vars x x_1 ... x x_2 ...) #false]
  [(unique-vars x x_1 ...) (unique-vars x_1 ...)])

(module+ test
  (test-results))
```

Submodules delegate the tests to where they belong and they allow us to document functions by example.

Sadly, our language definition cannot use the `unique-vars` metafunction. (In order to define the metafunction, we first need to define the language.)

Fortunately, language definitions can employ more than Kleene patterns:

```
(define-language Lambda
  (e ::=
     x
     (lambda (x_!_ ...) e)
     (e e ...))
  (x ::= variable-not-otherwise-mentioned))
```

`x_!_ ...` means `x` must differ from all other elements of this sequence

Here are two more metafunctions that use patterns in interesting ways:

```
; (subtract (x ...) x_1 ...) removes x_1 ... from (x ...)

(module+ test
  (test-equal (term (subtract (x y z x) x z)) (term (y))))

(define-metafunction Lambda
  subtract : (x ...) x ... -> (x ...)
  [(subtract (x ...)) (x ...)]
  [(subtract (x ...) x_1 x_2 ...)
   (subtract (subtract1 (x ...) x_1) x_2 ...)])

; (subtract1 (x ...) x_1) removes x_1  from (x ...)
(module+ test
  (test-equal (term (subtract1 (x y z x) x)) (term (y z))))
```

```
(define-metafunction Lambda
  subtract1 : (x ...) x -> (x ...)
  [(subtract1 (x_1 ... x x_2 ...) x)
   (x_1 ... x_2new ...)
   (where (x_2new ...) (subtract1 (x_2 ...) x))
   (where #false (in x (x_1 ...)))]
  [(subtract1 (x ...) x_1) (x ...)])

(define-metafunction Lambda
  in : x (x ...) -> boolean
  [(in x (x_1 ... x x_2 ...)) #true]
  [(in x (x_1 ...)) #false])
```

### 2.2.3 Developing a Language 2

One of the first things a language designer ought to specify is *scope*. People often do so with a free-variables function that specifies which language constructs bind and which ones don't:

```
; (fv e) computes the sequence of free variables of e
; a variable occurrence of x is free in e
; if no (lambda (... x ...) ...) dominates its occurrence

(module+ test
  (test-equal (term (fv x)) (term (x)))
  (test-equal (term (fv (lambda (x) x))) (term ()))
  (test-equal (term (fv (lambda (x) (y z x)))) (term (y z))))

(define-metafunction Lambda
  fv : e -> (x ...)
  [(fv x) (x)]
  [(fv (lambda (x ...) e))
   (subtract (x_e ...) x ...)
   (where (x_e ...) (fv e))]
  [(fv (e_f e_a ...))
   (x_f ... x_a ... ...)
   (where (x_f ...) (fv e_f))
   (where ((x_a ...) ...) ((fv e_a) ...))])
```

You may know it as the de Bruijn index representation.

The best approach is to specify an $\alpha$ equivalence relation, that is, the relation that virtually eliminates variables from phrases and replaces them with arrows to their declarations. In the world of lambda calculus-based languages, this transformation is often a part of the compiler, the so-called static-distance phase.

The function is a good example of accumulator-functions in Redex:

```
; (sd e) computes the static distance version of e

(define-extended-language SD Lambda
  (e ::= .... (K n))
  (n ::= natural))

(define sd1 (term (K 1)))
(define sd2 (term 1))

(define SD? (redex-match? SD e))

(module+ test
  (test-equal (SD? sd1) #true))
```

We have to add a means to the language to say "arrow back to the variable declaration." We do *not* edit the language definition but *extend* the language definition instead.

```
(define-metafunction SD
  sd : e -> e
  [(sd e_1) (sd/a e_1 ())])

(module+ test
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
  (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
              (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
  (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
              (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
  (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
              (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

(define-metafunction SD
  sd/a : e ((x ...) ...) -> e
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
  [(sd/a (lambda (x ...) e_1) (e_rest ...))
   (lambda () (sd/a e_1 ((x ...) e_rest ...)))]
  [(sd/a (e_fun e_arg ...) (e_rib ...))
   ((sd/a e_fun (e_rib ...)) (sd/a e_arg (e_rib ...)) ...)]
  [(sd/a e_1 any)
   ; a free variable is left alone
   e_1])
```

Now $\alpha$ equivalence is straightforward:

```
; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

(define-extended-language Lambda/n Lambda
  (e ::= .... n)
  (n ::= natural))

(define in-Lambda/n? (redex-match? Lambda/n e))

(module+ test
  (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
  (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
  (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

(define-metafunction SD
  =α : e e -> boolean
  [(=α e_1 e_2) ,(equal? (term (sd e_1)) (term (sd e_2)))])

(define (=α/racket x y) (term (=α ,x ,y)))
```

### 2.2.4 Extending a Language: `any`

Suppose we wish to extend `Lambda` with `if` and Booleans, like this:

```
(define-extended-language SD Lambda
  (e ::= ....
     true
     false
     (if e e e)))
```

Guess what? `(term (fv (lambda (x y) (if x y false))))` doesn't work because `false` and `if` are not covered.

We want metafunctions that are as generic as possible for computing such notions as free variable sequences, static distance, and alpha equivalences.

Redex contracts come with `any` and Redex patterns really are over Racket's S-expressions. This definition now works for extensions that don't add binders:

```
(module+ test
  (test-equal (SD? sd1) #true))

(define-metafunction SD
  sd : any -> any
```

```
  [(sd any_1) (sd/a any_1 ())])

(module+ test
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
  (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
              (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
  (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
              (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
  (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
              (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

(define-metafunction SD
  sd/a : any ((x ...) ...) -> any
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
  [(sd/a (lambda (x ...) any_1) (any_rest ...))
   (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
  [(sd/a (any_fun any_arg ...) (any_rib ...))
   ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
  [(sd/a any_1 any)
   ; free variable, constant, etc
   any_1])
```

### 2.2.5 Substitution

The last thing we need is substitution, because it *is* the syntactic equivalent of function application. We define it with *any* having future extensions in mind.

```
; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hy-
gienically)

(module+ test
  (test-equal (term (subst ([1 x][2 y]) x)) 1)
  (test-equal (term (subst ([1 x][2 y]) y)) 2)
  (test-equal (term (subst ([1 x][2 y]) z)) (term z))
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
              (term (lambda (z w) (1 2))))
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
              (term (lambda (z w) (lambda (x) (x 2))))
              #:equiv =α/racket)
```

```
  (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
              (term ((lambda (x) (1 x)) 2))
              #:equiv =α/racket))
```

```
(define-metafunction Lambda
  subst : ((any x) ...) any -> any
  [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
  [(subst [(any_1 x_1) ...]  x) x]
  [(subst [(any_1 x_1) ...]  (lambda (x ...) any_body))
   (lambda (x_new ...)
     (subst ((any_1 x_1) ...)
            (subst-raw ((x_new x) ...) any_body)))
   (where  (x_new ...)  ,(variables-not-in (term any_body) (term (x ...))))]
  [(subst [(any_1 x_1) ...]  (any ...)) ((subst [(any_1 x_1) ...]  any) ...)]
  [(subst [(any_1 x_1) ...]  any_*) any_*])

(define-metafunction Lambda
  subst-raw : ((x x) ...) any -> any
  [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
  [(subst-raw ((x_n1 x_o1) ...)  x) x]
  [(subst-raw ((x_n1 x_o1) ...)  (lambda (x ...) any))
   (lambda (x ...) (subst-raw ((x_n1 x_o1) ...)  any))]
  [(subst-raw [(any_1 x_1) ...]  (any ...))
   ((subst-raw [(any_1 x_1) ...]  any) ...)]
  [(subst-raw [(any_1 x_1) ...]  any_*) any_*])
```

## 2.3 Lab Designing Metafunctions

**Goals**
— developing meta-functions
— discovering Redex patterns

The following exercises refer to several definitions found in, and exported from, §2.13 “"common.rkt"”. You may either copy these definitions into your file or add the following `require` statement to the top of your file:

```
(require "common.rkt")
```

### Exercises

**Exercise** 1. Design `bv`. The metafunction determines the bound variables in a `Lambda` expression. A variable `x` is *bound in* `e_Lambda` if `x` occurs in a `lambda`-parameter list in `e_Lambda`. ▮

**Exercise** 2. Design `lookup`. The metafunction consumes a variable and an environment. It determines the leftmost expression associated with the variable; otherwise it produces `#false`.

Here is the definition of *environment*:

```
(define-extended-language Env Lambda
  (e ::= .... natural)
  (env ::= ((x e) ...)))
```

The language extension also adds numbers of the sub-language of expressions.

Before you get started, make sure you can create examples of environments and confirm their well-formedness. ▮

**Exercise** 3. Develop the metafunction `let`, which extends the language with a notational shorthand, also known as syntactic sugar.

Once you have this metafunction, you can write expressions such as

```
(term
  (let ((x (lambda (a b c) a))
        (y (lambda (x) x)))
    (x y y y)))
```

Like Racket's `let`, the function elaborates surface syntax into core syntax:

```
(term
  ((lambda (x y) (x y y y))
   (lambda (a b c) a)
   (lambda (x) x)))
```

Since this elaboration happens as the term is constructed, all other metafunctions work as expected on this extended syntax. For example,

```
(term
 (fv
  (let ((x (lambda (a b c) a))
        (y (lambda (x) x)))
    (x y y y))))
```

or

```
(term
 (bv
  (let ((x (lambda (a b c) a))
        (y (lambda (x) x)))
    (x y y y))))
```

produces the expected results. What are those? ▮

## 2.4 Reductions and Semantics

**Goals**
— extend languages with concepts needed for reduction relations
— developing reduction relations
— defining a semantics
— testing against a language

**Note** These notes deal with the $\lambda\beta$ calculus, specifically its reduction system.

| notation | meaning |
|---|---|
| `x` | basic notion of reduction, without properties |
| `-->x` | one-step reduction, generated from `x`, compatible with syntactic constructions |
| `-->>x` | reduction, generated from `-->x`, transitive here also reflexive |
| `=x` | "calculus", generated from `-->x`, symmetric, transitive, reflexive |

### 2.4.1 Contexts, Values

The logical way of generating an equivalence (or reduction) relation over terms uses through inductive inference rules that make the relation compatible with all syntactic constructions.

An alternative and equivalent method is to introduce the notion of a context and to use it to generate the reduction relation (or equivalence) from the notion of reduction:

```
(require "common.rkt")

(define-extended-language Lambda-calculus Lambda
  (e ::= .... n)
  (n ::= natural)
  (v ::= (lambda (x ...) e))

  ; a context is an expression with one hole in lieu of a sub-
expression
  (C ::=
     hole
```

```
    (e ... C e ...)
    (lambda (x_!_ ...) C)))

(define Context? (redex-match? Lambda-calculus C))

(module+ test
  (define C1 (term ((lambda (x y) x) hole 1)))
  (define C2 (term ((lambda (x y) hole) 0 1)))
  (test-equal (Context? C1) #true)
  (test-equal (Context? C2) #true))
```

Filling the hole of context with an expression yields an expression:

```
(module+ test
  (define e1 (term (in-hole ,C1 1)))
  (define e2 (term (in-hole ,C2 x)))

  (test-equal (in-Lambda/n? e1) #true)
  (test-equal (in-Lambda/n? e2) #true))
```

What does filling the hole of a context with a context yield?

### 2.4.2 Reduction Relations

Developing a reduction relation is like developing a function. Work through examples first. A reduction relation does not have to be a function, meaning it may reduce one and the same term to distinct terms.

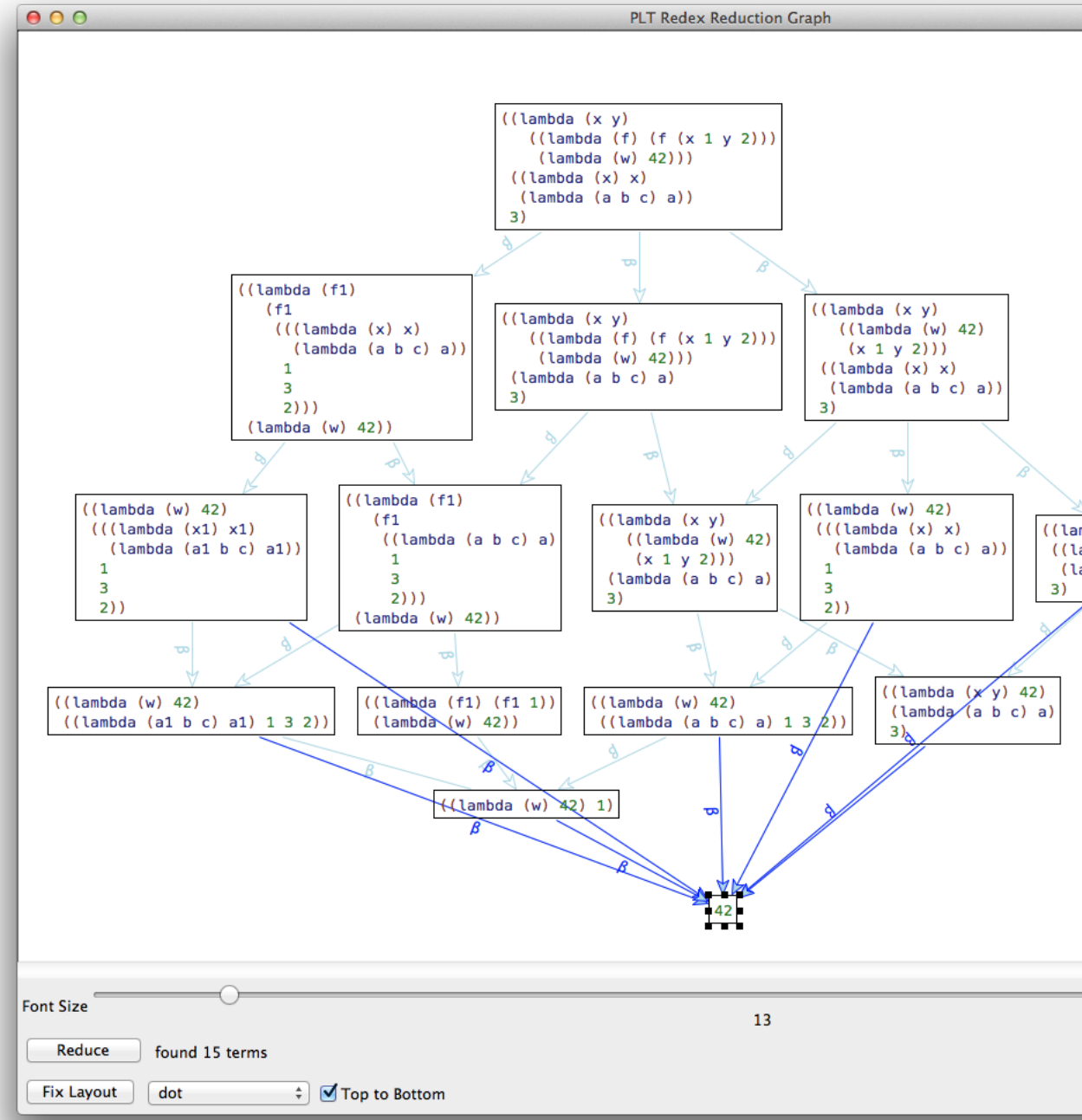
```
; the λβ calculus, reductions only
(module+ test
  ; does the one-step reduction reduce both β redexes?
  (test--> -->β
           #:equiv =α/racket
           (term ((lambda (x) ((lambda (y) y) x)) z))
           (term ((lambda (x) x) z))
           (term ((lambda (y) y) z)))

  ; does the full reduction relation reduce all redexes?
  (test-->> -->β
            (term ((lambda (x y) (x 1 y 2))
                   (lambda (a b c) a)
                   3))
            1))
```

```
(define -->β
  (reduction-relation
   Lambda-calculus
   (--> (in-hole C ((lambda (x_1 ..._n) e) e_1 ..._n))
        (in-hole C (subst ([e_1 x_1] ...) e)))))
```

With `traces` we can visualize reduction graphs:

```
(traces -->β
        (term ((lambda (x y)
                 ((lambda (f) (f (x 1 y 2)))
                  (lambda (w) 42)))
               ((lambda (x) x) (lambda (a b c) a))
               3)))
```


PLT Redex Reduction Graph
((lambda (x y)
((lambda (f) (f (x 1 y 2)))
(lambda (w) 42)))
((lambda (x) x)
(lambda (a b c) a))
3)
((lambda (f1)
(f1
(((lambda (x) x)
(lambda (a b c) a))
1
3
2)))
(lambda (w) 42))
((lambda (x y)
((lambda (f) (f (x 1 y 2)))
(lambda (w) 42)))
(lambda (a b c) a)
3)
((lambda (x y)
((lambda (w) 42)
(x 1 y 2)))
((lambda (x) x)
(lambda (a b c) a))
3)
((lambda (w) 42)
(((lambda (x1) x1)
(lambda (a1 b c) a1))
1
3
2))
((lambda (f1)
(f1
((lambda (a b c) a)
1
3
2)))
(lambda (w) 42))
((lambda (x y)
((lambda (w) 42)
(x 1 y 2)))
(lambda (a b c) a)
3)
((lambda (w) 42)
(((lambda (x) x)
(lambda (a b c) a))
1
3
2))
((lambda (w) 42)
((lambda (a1 b c) a1) 1 3 2))
((lambda (f1) (f1 1))
(lambda (w) 42))
((lambda (w) 42)
((lambda (a b c) a) 1 3 2))
((lambda (x y) 42)
(lambda (a b c) a)
3)
((lambda (w) 42) 1)
42
β
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13
Reduce
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Defining the call-by-value calculus requires just a small change to the reduction rule:

```
(define -->βv
  (reduction-relation
   Lambda-calculus
   (--> (in-hole C ((lambda (x_1 ..._n) e) v_1 ..._n))
        (in-hole C (subst ([v_1 x_1] ...) e)))))
```

Let's compare traces for the same term. You do get the same result but significantly fewer intermediate terms. Why?

### 2.4.3 Semantics

First we need a standard reduction relation. The key is to define the path to the leftmost-outermost redex, which can again be done via contexts. Here are the relevant definitions for the by-value reduction relation:

```
(define-extended-language Standard Lambda-calculus
  (v ::= n (lambda (x ...) e))
  (E ::=
     hole
     (v ... E e ...)))

(module+ test
  (define t0
    (term
     ((lambda (x y) (x y))
      ((lambda (x) x) (lambda (x) x))
      ((lambda (x) x) 5))))
  (define t0-one-step
    (term
     ((lambda (x y) (x y))
      (lambda (x) x)
      ((lambda (x) x) 5))))

  ; yields only one term, leftmost-outermost
  (test--> s->βv t0 t0-one-step)
  ; but the transitive closure drives it to 5
  (test-->> s->βv t0 5))

(define s->βv
  (reduction-relation
   Standard
   (--> (in-hole E ((lambda (x_1 ..._n) e) v_1 ..._n))
        (in-hole E (subst ((v_1 x_1) ...) e)))))
```

Note the tests-first development of the relation.

Now we can define the semantics function:

```
(module+ test
  (test-equal (term (eval-value ,t0)) 5)
  (test-equal (term (eval-value ,t0-one-step)) 5)

  (define t1
    (term ((lambda (x) x) (lambda (x) x))))
  (test-equal (lambda? t1) #true)
  (test-equal (redex-match? Standard e t1) #true)
  (test-equal (term (eval-value ,t1)) 'closure))

(define-metafunction Standard
  eval-value : e -> v or closure
  [(eval-value e) any_1 (where any_1 (run-value e))])

(define-metafunction Standard
  run-value : e -> v or closure
  [(run-value n) n]
  [(run-value v) closure]
  [(run-value e)
   (run-value e_again)
   ; (v) means that we expect s->βv to be a function
   (where (e_again) ,(apply-reduction-relation s->βv (term e)))])
```

The key is the stepper-loop, which applies the Racket function `apply-reduction-relation` repeatedly until it yields a value.

### 2.4.4 What are Models

Good models of programming languages are like Newtonian models of how you drive a car. As long as your speed is within a reasonable interval, the model accurately predicts how your car behaves. Similarly, as long as your terms are within a reasonable subset (the model's language), the evaluator of the model and the evaluator of the language ought to agree.

For Racket you set up an evaluator for the language like this:

```
(define-namespace-anchor A)
(define N (namespace-anchor->namespace A))
; Lambda.e -> Number or 'closure or exn
(define (racket-evaluator t0)
  (define result
```

```
    (with-handlers ((exn:fail? values))
      (eval t0 N)))
  (cond
    [(number? result) result]
    [(procedure? result) (term closure)]
    [else (make-exn "hello world" (current-continuation-marks))]))
```

The details don't matter.

So the theorem is this:

```
(define-metafunction Standard
  theorem:racket=eval-value : e -> boolean
  [(theorem:racket=eval-value e)
   ,(letrec ([rr (racket-evaluator (term e))]
             [vr (term (run-value e))])
      (cond
        [(and (exn? rr) (eq? (term stuck) vr))
         #true]
        [(exn? rr) #false]
        [(eq? (term stuck) vr) #false]
        [else (equal? vr rr)]))])
```

We formulate it as a meta-function and test it on some terms:

```
(module+ test
  (test-equal (term (racket=eval-value ,t0)) #true)
  (test-equal (term (racket=eval-value ,t0-one-step)) #true)
  (test-equal (term (racket=eval-value ,t1)) #true))
```

The real test comes with random testing:

```
(redex-check Standard e (term (theorem:racket=eval-value e)))
```

And now it's time to discover.

## 2.5 Lab Designing Reductions

**Goals**
— developing reductions
— semantics

The following exercises refer to several definitions found in, and exported from, §2.13 "`"common.rkt"`". You may either copy these definitions into your file or add the following `require` statement to the top of your file:

```
(require "common.rkt")
```

Also `require` §2.14 "`"close.rkt"`" for the `fv` function.

You also want to copy the following definitions into your drracket:

```
(define-extended-language Lambda-η Lambda
  (e ::= .... n)
  (n ::= natural)
  (C ::=
     hole
     (e ... C e ...)
     (lambda (x_!_ ...) C))
  (v ::=
     n
     (lambda (x ...) e)))

(define -->β
  (reduction-relation
   Lambda-η
   (--> (in-hole C ((lambda (x_1 ..._n) e) e_1 ..._n))
        (in-hole C (subst ([e_1 x_1] ...) e))
        β)))

(define lambda? (redex-match? Lambda-calculus e))
```

Consider equipping the one-step reduction relation with tests.

### Exercises

**Exercise** 4. Develop a $\beta\eta$ reduction relation for `Lambda-η`.

Find a term that contains both a $\beta$- and an $\eta$-redex. Formulate a Redex test that validates this claim. Also use `trace` to graphically validate the claim.

Develop the $\beta$ and $\beta\eta$ standard reduction relations. **Hint** Look up `extend-reduction-relation` to save some work.

Use the standard reduction relations to formulate a semantics for both variants. The above test case, reformulated for the standard reduction, *must* fail. Why? **Note** The semantics for $\beta\eta$ requires some experimentation. Justify your non-standard definition of the `run` function.

The $\beta\eta$ semantics is equivalent to the $\beta$ variant. Formulate this theorem as a metafunction. Use `redex-check` to test your theorem.

**Note** Why does it make no sense to add $\eta$ to this system? ▌

**Exercise** 5. Extend the by-value language with an addition operator.

Equip both the $\beta$v reduction system and the $\beta$v standard reduction with rules that assign addition the usual semantics. Finally define a semantics functions for this language.

**Hint** Your rules need to escape to Racket and use its addition operator. ▮

## 2.6 Types and Property Testing

**Goals**
— typed languages
— developing type judgments
— subject reduction

### 2.6.1 Types

Here is a typed variant of the Lambda language:

```
(define-language TLambda
  (e ::=
     n
     +
     x
     (lambda ((x_!_ t) ...) e)
     (e e ...))
  (t ::=
     int
     (t ... -> t))
  (x ::= variable-not-otherwise-mentioned))

(define lambda? (redex-match? TLambda e))

(define e1
  (term (lambda ((x int) (f (int -> int))) (+ (f (f x)) (f x)))))
(define e2
  (term (lambda ((x int) (f ((int -> int) -> int))) (f x))))
(define e3
  (term (lambda ((x int) (x (int -> int))) x)))

(module+ test
  (test-equal (lambda? e1) #true)
  (test-equal (lambda? e2) #true)
  (test-equal (in-TLambda? e3) #false))
```

### 2.6.2 Developing Type Judgments

Like metafunctions and reduction relations, type judgments are developed by working out examples, formulating tests, and then articulating the judgment rules:

```
; (⊢ Γ e t) - the usual type judgment for an LC language

(define-extended-language TLambda-tc TLambda
  (Γ ::= ((x t) ...)))

(module+ test
  (test-equal (judgment-holds (⊢ () ,e1 (int (int -> int) -> int))) #true)
  (test-equal (judgment-holds (⊢ () ,e2 t)) #false)
  (displayln  (judgment-holds (⊢ () ,e1 t) t))
  (displayln  (judgment-holds (⊢ () ,e2 t) t)))

(define-judgment-form TLambda-tc
  #:mode (⊢ I I O)
  #:contract (⊢ Γ e t)
  [----------------------- "number"
   (⊢ Γ n int)]

  [----------------------- "+"
   (⊢ Γ + (int int -> int))]

  [----------------------- "variable"
   (⊢ Γ x (lookup Γ x))]

  [(⊢ (extend Γ (x_1 t_1) ...) e t)
   ------------------------------------------------- "lambda"
   (⊢ Γ (lambda ((x_1 t_1) ...) e) (t_1 ... -> t))]

  [(⊢ Γ e_1 (t_2 ... -> t))
   (⊢ Γ e_2 t_2) ...
   ------------------------------------------------- "application"
   (⊢ Γ (e_1 e_2 ...) t)])
```

Here are the necessary auxiliary functions:

```
; (extend Γ (x t) ...) add (x t) to Γ so that x is found before
other x-s
(module+ test
  (test-equal (term (extend () (x int))) (term ((x int)))))

(define-metafunction TLambda-tc
```

```
  extend : Γ (x t) ... -> Γ
  [(extend ((x_Γ t_Γ) ...) (x t) ...) ((x t) ...(x_Γ t_Γ) ...)])

; (lookup Γ x) retrieves x's type from Γ
(module+ test
  (test-equal (term (lookup ((x int) (x (int -> int)) (y int)) x)) (term int))
  (test-equal (term (lookup ((x int) (x (int -> int)) (y int)) y)) (term int)))

(define-metafunction TLambda-tc
  lookup : Γ x -> t
  [(lookup ((x_1 t_1) ... (x t) (x_2 t_2) ...) x)
   t
   (side-condition (not (member (term x) (term (x_1 ...)))))]
  [(lookup any_1 any_2) ,(error 'lookup "not found:
~e" (term x))])
```

### 2.6.3 Subjection Reduction

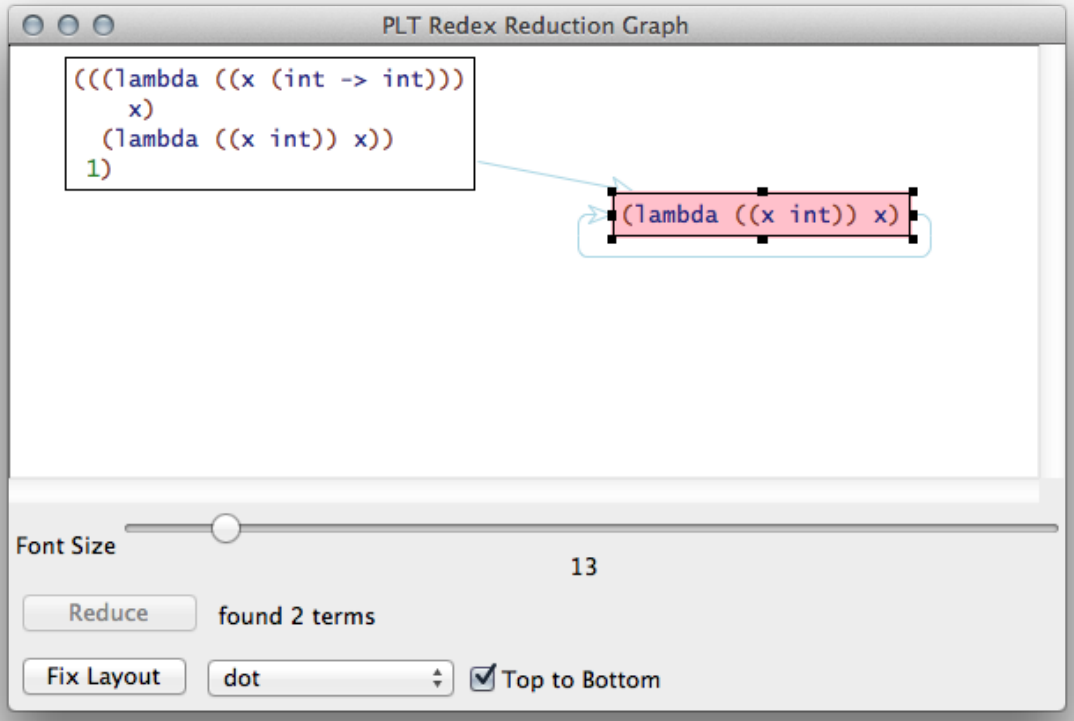
Let's say we define a truly broken (standard) reduction relation for `TLambda`:

```
(define ->
  (reduction-relation
   TLambda
   #:domain e
   (--> e (lambda ((x int)) x))))
```

With `trace`, we can quickly see that paths in almost any term's reduction graph do not preserve types:

```
(traces ->
      (term (((lambda ((x (int -> int))) x) (lambda ((x int)) x)) 1))
      #:pred (lambda (e) (judgment-holds (⊢ () ,e int))))
```

The `#:pred` keyword argument supplies a Racket function that judges whether the intermediate expression type checks, using our type judgment from above.




For simple "type systems," `redex-check` can be used to test a true subject reduction statement:

```
(redex-check TLambda
             e
             (implies (judgment-holds (⊢ () e int))
                      (judgment-holds (⊢ () (eval-value e) int)))
             #:attempts 3)
```

## 2.7 Lab Type Checking

**Goals**
— subject reduction testing with `trace`
— typing judgments

The following exercises refer to several definitions found in, and exported from, §2.13 “"common.rkt"”. You may either copy these definitions into your file or add the following `require` statement to the top of your file:

```
(require "common.rkt")
```

In addition to §2.13 “"common.rkt"”, you also want to `require` §2.15 “"tc-common.rkt"” for this lab. Furthermore, if you copy code from §2.6 “Types and Property Testing”, make sure to copy the tests and to adapt the tests as you develop the machines.

#### Exercises

**Exercise** 6. Develop a reduction system for which the `trace` expression from the lecture preserves types

```
(module+ test
  (traces ->
          (term (((lambda ((x (int -> int))) x) (lambda ((x int)) x)) 1))
          #:pred (lambda (e)
                   (judgment-holds (⊢ () ,e int)))))
```

▮

**Exercise** 7. Extend `TLambda` with syntax for the following:

- additional numeric operators, say, multiplication, subtraction, and division;
- `let` expressions;
- Boolean constants plus strict and and or operators as well as a branching construct;
- lists, specifically constructors and selectors (de-constructors);
- explicitly recursive function definitions.

Completing the above list is an ambitious undertaking, but do try to complete at least two or three of these tasks. ▮

## 2.8 Imperative Extensions

**Goals**
— revise the language for assignment statements
— a standard reduction system for expression-store tuples
— revise the language for raising exceptions
— a general reduction system for exceptions

### 2.8.1 Variable Assignment

Let's add variable assignments to our language:

```
(define-extended-language Assignments Lambda
  (e ::= .... n + (void) (set! x e))
  (n ::= natural))
```

This makes it like Racket, Scheme and Lisp, but unlike ML where you can mutate only data structure (one-slot records in SML and slots in arbitrary records in OCaml.

For writing programs in this world, you also want blocks and local declarations. We add a task-specific `let` expression:

```
; (let ((x_1 x_2) ...) e_1 e_2) binds the current value of x_2 to
x_1,
; evaluates e_1, throws away its value, and finally evaluates e_2
(define-metafunction Assignments
  let : ((x e) ...) e e -> e
  [(let ([x_lhs e_rhs] ...) e_1 e_2)
   ((lambda (x_lhs ...)
      ((lambda (x_dummy) e_2) e_1))
    e_rhs ...)
   (where (x_dummy) ,(variables-not-in (term (e_1 e_2)) '(dummy)))])
```

Here are some sample programs:

```
(define e1
  (term
   (lambda (x)
     (lambda (y)
       (let ([tmp x])
         (set! x (+ y 1))
         tmp)))))

(define p-1 (term ((,e1 1) 2)))

(define e2
  (term
   ((lambda (x)
      (let ([tmp x])
        (set! x y)
        tmp))
    (let ([tmp-z z])
      (set! z (+ z 1))
```

```
      (let ([tmp-y y])
        (set! y tmp-z)
        tmp-y)))))

(define p-2
  (term ((lambda (y) ((lambda (z) ,e2) 1)) 2)))
```

How do they behave?

For a *standard reduction relation*, we need both evaluation contexts *and* a table that keeps track of the current value of variables:

```
(define-extended-language Assignments-s Assignments
  (E ::= hole (v ... E e ...) (set! x E))
  (σ ::= ((x v) ...))
  (v ::= n + (void) (lambda (x ...) e)))

; (extend σ x v) adds (x v) to σ
(define-metafunction Assignments-s
  extend : σ (x ...) (any ...) -> σ
  [(extend ((x any) ...) (x_1 ...) (any_1 ...)) ((x_1 any_1) ... (x any) ...)])

; ---------------------------------------------------
; (lookup Γ x) retrieves x's type from Γ
(define-metafunction Assignments-s
  lookup : any x -> any
  [(lookup ((x_1 any_1) ... (x any_t) (x_2 any_2) ...) x)
   any_t
   (side-condition (not (member (term x) (term (x_1 ...)))))]
  [(lookup any_1 any_2)
   ,(error 'lookup "not found: ~e in: ~e" (term x) (term any_2))])
```

Extending this table and looking up values in it, is a routine matter by now.

Here is the standard reduction relation:

```
(define s->βs
  (reduction-relation
   Assignments-s
   #:domain (e σ)
   (--> [(in-hole E x) σ]
        [(in-hole E (lookup σ x)) σ])
   (--> [(in-hole E (set! x v)) σ]
        [(in-hole E (void)) (extend σ (x) (v))])
   (--> [(in-hole E (+ n_1 n_2)) σ]
        [(in-hole E ,(+ (term n_1) (term n_2))) σ])
```

```
    (--> [(in-hole E ((lambda (x ..._n) e) v ..._n)) σ]
         [(in-hole E e) (extend σ (x_new ...) (v ...))]
         (where (x_new ...) ,(variables-not-
in (term σ) (term (x ...)))))))
```

The question is what the corresponding calculus looks like. See §2.9 "**Lab** Contexts and Stores".

**This use of the standard reduction relation is common because most researchers *do not need* the calculus. Instead they define such a relation and consider it a semantics.**

The semantics is a function, however, that maps programs to the final answers and possibly extracts pieces from the store.

```
(module+ test
  (test-equal (term (eval-assignments ,p-1)) 1)
  (test-equal (term (eval-assignments ,p-2)) 2)
  (test-equal (term (eval-assignments ,p-c)) (term closure)))

(define-metafunction Assignments-s
  eval-assignments : e -> v or closure
  [(eval-assignments e) (run-assignments (e ()))])

(define-metafunction Assignments-s
  run-assignments : (e σ) -> v or closure
  [(run-assignments (n σ)) n]
  [(run-assignments (v σ)) closure]
  [(run-assignments any_1)
   (run-assignments any_again)
   (where (any_again) ,(apply-reduction-relation s-
>βs (term any_1)))]
  [(run-assignments any) stuck])
```

### 2.8.2 Raising Exceptions

When non-local control operators come such as ML's exceptions come into play, reductions become (evaluation-) context-sensitive.

Here is a language with a simple construct for raising exceptions:

```
(define-extended-language Exceptions Lambda
  (e ::= .... n + (raise e))
  (n ::= integer))
```

**History** This form of exception was actually introduced into Lisp as the `catch` and `throw`

combination (contrary to some statements in Turing-award announcements).

Try to figure out what these expressions ought to compute:

```
(define c1
  (term
   ((lambda (x)
      (+ 1 (raise (+ 1 x))))
    0)))

(define c2
  (term
   (lambda (y)
     ((lambda (x)
        (+ 1 (raise (+ (raise -1) x))))
      0))))
```

A calculus of exceptions needs both arbitrary term contexts and evaluation contexts:

```
(define-extended-language Exceptions-s Exceptions
  (C ::= hole (e ... C e ...) (lambda (x ...) C) (raise C))
  (E ::= hole (v ... E e ...) (raise E))
  (v ::= n + (lambda (x ...) e)))
```

The key insight is that an exception-raising construct erases any surrounding evaluation context, regardless of where it shows up:

```
(module+ test
  (test-->> ->βc c1 (term (raise 1)))
  (test-->> ->βc c2 (term (lambda (y) (raise -1)))))

(define ->βc
  (reduction-relation
   Exceptions-s
   (--> (in-hole C (in-hole E (raise v)))
        (in-hole C (raise v))
        (where #false ,(equal? (term E) (term hole)))
        ζ)
   (--> (in-hole C (+ n_1 n_2))
        (in-hole C ,(+ (term n_1) (term n_2)))
        +)
   (--> (in-hole C ((lambda (x_1 ..._n) e) v_1 ..._n))
        (in-hole C (subst ([v_1 x_1] ...) e))
        β_v)))
```

The question is what a standard reduction relation for such a calculus looks like. See §2.9 “**Lab** Contexts and Stores”.

## 2.9 Lab Contexts and Stores

**Goals**
— develop a general reduction system for Lambda with assignments
— develop a standard reduction system for Lambda with exceptions

The following exercises refer to several definitions found in, and exported from, §2.13 "`"common.rkt"`". You may either copy these definitions into your file or add the following `require` statement to the top of your file:

```
(require "common.rkt")
```

Also require §2.16 "`"extend-lookup.rkt"`". Feel free to copy code from §2.8 "Imperative Extensions" but make sure to add tests.

### Exercises

The exercises this morning are puzzles. Try your hands on them, but when you feel stuck, don't hesitate to request help.

**Exercise** 8. Develop a reduction relation for assignment statements. Add a `letrec` syntax to the language like this:

```
(define-extended-language ImperativeCalculus Assignments
  (e ::= .... (letrec ((x v) ...) e)))
```

A `letrec` mutually recursively binds the variables `x ...` to the values `v ...` and in `e`. The addition of `letrec` internalizes the store into the language. Adapt the existing relations.

Develop terms that one-step reduce in several different directions via reductions that model assignment and/or variable derefences. Use `trace` graphs to demonstrate the idea.

**Note** This calculus has naturally separated mini-heaps, but your system must extrude the scope of these heaps on occasion (when values are returned) and merge them. ▮

**Exercise** 9. Develop a standard reduction system and a semantics for exceptions.

**Note** You need to use evaluation contexts for two distinct purposes. ▮

**Exercise** 10. Develop a semantics of for a control operator such as `callcc`.

**Request** Check with one of us before you embark on this project. We want to make sure that (1) the operator isn't too difficult and (2) not to easy to implement. We are also available for hints. ▮

## 2.10 Abstract Machines

**Goals**

— why these three machines: CC machine, CK machine, CEK machine
— theorems connecting the machines, theorems for debugging
— equivalence theorems

### 2.10.1 CC Machine

**Observation** $\beta$ and $\beta$_v redexes often take place repeatedly in the same evaluation context. On occasion they just add more layers (inside the hole) to the evaluation context. Let's separate the in-focus expression from the evaluation context. Historically the two have been called *control string* (C) and *control context* (C).

```
(define-extended-language Lambda/v Lambda
  (e ::= .... n +)
  (n ::= integer)
  (v ::= n + (lambda (x ...) e)))

(define vv? (redex-match? Lambda/v e))

(define e0
  (term ((lambda (x) x) 0)))

(define e1
  (term ((lambda (x y) x) 1 2)))

(module+ test
  (test-equal (vv? e1) #true)
  (test-equal (vv? e0) #true))

; -----------------------------------------------------
; the CC machine: keep contexts and expression-in-focus apart

(define-extended-language CC Lambda/v
  (E ::=
     hole
     ; Note right to left evaluation of application
     (e ... E v ...)))

(module+ test
  (test-->> -->cc (term [,e0 hole]) (term [0 hole]))
  (test-->> -->cc (term [,e1 hole]) (term [1 hole])))

(define -->cc
  (reduction-relation
   CC
   #:domain (e E)
```

```
   (--> [(lambda (x ..._n) e)
         (in-hole E (hole v ..._n))]
        [(subst ([v x] ...) e) E]
        CC-β_v)
   (--> [+
         (in-hole E (hole n_1 n_2))]
        [,(+ (term n_1) (term n_2)) E]
        CC-+)
   (--> [(e_1 ...) E]
        [e_last (in-hole E (e_1others ... hole))]
        (where (e_1others ... e_last) (e_1 ...))
        CC-push)
   (--> [v      (in-hole E (e ... hole v_1 ...))]
        [e_last (in-hole E (e_prefix ... hole v v_1 ...))]
        (where (e_prefix ... e_last) (e ...))
        CC-switch)))

(module+ test
  (test-equal (term (eval-cc ,e0)) 0)
  (test-equal (term (eval-cc ,e1)) 1))

(define-metafunction Lambda/v
  eval-cc : e -> v or closure or stuck
  [(eval-cc e) (run-cc [e hole])])

(define-metafunction CC
  run-cc : (e E) -> v or closure or stuck
  [(run-cc (n hole)) n]
  [(run-cc (v hole)) closure]
  [(run-cc any_1)
   (run-cc (e_again E_again))
   (where ((e_again E_again)) ,(apply-reduction-
relation -->cc (term any_1)))]
  [(run-cc any) stuck])
```

### 2.10.2 The CK Machine

**Observation** The evaluation context of the CC machine behaves exactly like a control stack. Let's represent it as such.

**General Idea** The general idea is to show how valuable it is to reconsider data representations in PL, and how easy it is to do so in Redex.

```
(define-extended-language CK Lambda/v
  ; Note encode context as stack (left is top)
```

```
  (k ::= ((app [v ...] [e ...]) ...)))

(module+ test
  (test-->> -->ck (term [,e0 ()]) (term [0 ()]))
  (test-->> -->ck (term [,e1 ()]) (term [1 ()])))

(define -->ck
  (reduction-relation
   CK
   #:domain (e k)
   (--> [(lambda (x ..._n) e)
         ((app [v ..._n] []) (app any_v any_e) ...)]
        [(subst ([v x] ...) e)
         ((app any_v any_e) ...)]
        CK-β_v)
   (--> [+ ((app [n_1 n_2] []) (app any_v any_e) ...)]
        [,(+ (term n_1) (term n_2)) ((app any_v any_e) ...)]
        CK-+)
   (--> [(e_1 ...) (any_k ...)]
        [e_last ((app () (e_1others ...)) any_k ...)]
        (where (e_1others ... e_last) (e_1 ...))
        CK-push)
   (--> [v ((app (v_1 ...) (e ...)) any_k ...)]
        [e_last ((app (v v_1 ...) (e_prefix ...)) any_k ...)]
        (where (e_prefix ... e_last) (e ...))
        CK-switch)))

(module+ test
  (test-equal (term (eval-ck ,e0)) 0)
  (test-equal (term (eval-ck ,e1)) 1))

(define-metafunction Lambda/v
  eval-ck : e -> v or closure or stuck
  [(eval-ck e) (run-ck [e ()])])

(define-metafunction CK
  run-ck : (e k) -> v or closure or stuck
  [(run-ck (n ())) n]
  [(run-ck (v ())) closure]
  [(run-ck any_1)
   (run-ck (e_again k_again))
   (where ((e_again k_again)) ,(apply-reduction-
relation -->ck (term any_1)))]
  [(run-ck any) stuck])
```

### 2.10.3 The CC-CK Theorem

The two machines define the same evaluation function. Let's formulate this as a theorem and `redex-check` it.

**Note** When I prepared these notes, I found two mistakes in my machines.

```
(module+ test
  ; theorem:eval-ck=eval-cc
  (test-equal (term (theorem:eval-ck=eval-cc ,e0)) #true)
  (test-equal (term (theorem:eval-ck=eval-cc ,e1)) #true)

  ; NEXT: CEK vs CK
  (redex-check Lambda e (term (theorem:eval-ck=eval-cc e))
               #:attempts 24
               #:prepare (close-all-fv vv?)))

(define-metafunction Lambda/v
  theorem:eval-ck=eval-cc : e -> boolean
  [(theorem:eval-ck=eval-cc e)
   ,(equal? (term (eval-cc e)) (term (eval-ck e)))])
```

### 2.10.4 The CEK machine

**Observation** Substitution is an eager operation. It traverses the term kind of like machine does anyway when it searches for a redex. Why not combine the two by delaying substitution until needed? That's called an environment (E) in the contexts of machines (also see above).

**General Idea** Universal laziness is *not* a good idea. But the selective delay of operations—especially when operations can be merged—is a good thing.

```
(define-extended-language CEK Lambda/v
  (ρ ::= ((x c) ...))
  (c ::= (v ρ))
  (k ::= ((app [c ...] ρ [e ...]) ...)))

(module+ test
  (test-->> -->cek (term [,e0 () ()]) (term [0 () ()]))
  (test-->> -->cek (term [,e1 () ()]) (term [1 () ()])))

(define -->cek
  (reduction-relation
   CEK
   #:domain (e ρ k)
   (--> [x
```

```
          ((x_1 c_1) ... (x (v ρ)) (x_2 c_2) ...)
          ((app any_v any_r any_e) ...)]
         [v
          ρ
          ((app any_v any_r any_e) ...)]
         CEK-lookup)
   (--> [(lambda (x ..._n) e)
          (any_c ...)
          ((app [c ..._n] ρ []) (app any_v any_r any_e) ...)]
         [e
          ([x c] ... any_c ...)
          ((app any_v any_r any_e) ...)]
         CEK-β_v)
   (--> [+
          ρ
          ((app [n_1 n_2] []) (app any_v any_r any_e) ...)]
         [,(+ (term n_1) (term n_2))
          ()
          ((app any_v any_r any_e) ...)]
         CEK-+)
   (--> [(e_1 ...)
          ρ
          (any_k ...)]
         [e_last
          ρ
          ((app () ρ (e_1others ...)) any_k ...)]
         (where (e_1others ... e_last) (e_1 ...))
         CEK-push)
   (--> [v
          ρ
          ((app (c_1 ...) ρ_stack (e ...)) any_k ...)]
         [e_last
          ρ_stack
          ((app ((v ρ) c_1 ...) ρ_stack (e_prefix ...)) any_k ...)]
         (where (e_prefix ... e_last) (e ...))
         CEK-switch)))

(module+ test
  (test-equal (term (eval-cek ,e0)) 0)
  (test-equal (term (eval-cek ,e1)) 1))

(define-metafunction Lambda/v
  eval-cek : e -> v or closure or stuck
  [(eval-cek e) (run-cek [e () ()])])

(define-metafunction CEK
```

```
run-cek : (e ρ k) -> v or closure or stuck
[(run-cek (n ρ ())) n]
[(run-cek (v ρ ())) closure]
[(run-cek any_1)
 (run-cek (e_again ρ_again k_again))
 (where ((e_again ρ_again k_again))
        ,(apply-reduction-relation -->cek (term any_1)))]
[(run-cek any) stuck])
```

### 2.10.5 The CEK-CK Theorem

Again, the two machines define the *same semantics*. Here is the theorem.

```
(module+ test
  ; theorem:eval-ck=eval-cc
  (test-equal (term (theorem:eval-cek=eval-ck ,e0)) #true)
  (test-equal (term (theorem:eval-cek=eval-ck ,e1)) #true)

  ; NEXT: CEK vs CK
  (redex-check Lambda e (term (theorem:eval-cek=eval-ck e))
               #:attempts 24
               #:prepare (close-all-fv vv?)))

(define-metafunction Lambda/v
  theorem:eval-cek=eval-ck : e -> boolean
  [(theorem:eval-cek=eval-ck e)
   ,(equal? (term (eval-cek e)) (term (eval-ck e)))])
```

## 2.11 Lab Machine Transitions

**Goals**
— develop the CESK machine

The following exercises refer to several definitions found in, and exported from, §2.13 “`"common.rkt"`”. You may either copy these definitions into your file or add the following `require` statement to the top of your file:

```
(require "common.rkt")
```

In addition to §2.13 “`"common.rkt"`”, you also want to `require` §2.14 “`"close.rkt"`” for this lab. Furthermore, if you copy code from §2.10 “Abstract Machines”, make sure to copy the tests and to adapt the tests as you develop the machines.

### Exercises

**Exercise** 11. Equip the language with assignment statements and `void`:

```
(define-extended-language Assignments Lambda
  (e ::= .... n + (void) (set! x e))
  (n ::= natural))
```

Start with the CS reduction system and develop the CESK machine, re-tracing the above machine derivation. ▮

## 2.12 Abstracting Abstract Machines

David Van Horn presented his tutorial on Abstracting Abstract Machines in Redex.

## 2.13 "common.rkt"

```
#lang racket

;; basic definitions for the Redex Summer School 2015

(provide
 ;; Language
 Lambda

 ;; Any -> Boolean
 ;; is the given value in the expression language?
 lambda?

 ;; x (x ...) -> Boolean
 ;; (in x (x_1 ...)) determines whether x occurs in x_1 ...
 in

 ;; Any Any -> Boolean
 ;; (=α/racket e_1 e_2) determines whether e_1 is α-
equivalent to e_2
 ;; e_1, e_2 are in Lambda or extensions of Lambda that
 ;; do not introduce binding constructs beyond lambda
 =α/racket

 ;; ((Lambda x) ...) Lambda -> Lambda
 ;; (subs ((e_1 x_1) ...) e) substitures e_1 for x_1 ... in e
```

```
   ;; e_1, ... e are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   subst)

  ;; ---------------------------------------------------------------------
-------------
  (require redex)

  (define-language Lambda
    (e ::=
       x
       (lambda (x_!_ ...) e)
       (e e ...))
    (x ::= variable-not-otherwise-mentioned))

  (define lambda? (redex-match? Lambda e))

  (module+ test
    (define e1 (term y))
    (define e2 (term (lambda (y) y)))
    (define e3 (term (lambda (x y) y)))
    (define e4 (term (,e2 e3)))

    (test-equal (lambda? e1) #true)
    (test-equal (lambda? e2) #true)
    (test-equal (lambda? e3) #true)
    (test-equal (lambda? e4) #true)

    (define eb1 (term (lambda (x x) y)))
    (define eb2 (term (lambda (x y) 3)))

    (test-equal (lambda? eb1) #false)
    (test-equal (lambda? eb2) #false))

  ;; ---------------------------------------------------------------------
-------------
  ;; (in x x_1 ...) is x a member of (x_1 ...)?

  (module+ test
    (test-equal (term (in x (y z x y z))) #true)
    (test-equal (term (in x ())) #false)
    (test-equal (term (in x (y z w))) #false))

  (define-metafunction Lambda
    in : x (x ...) -> boolean
    [(in x (x_1 ... x x_2 ...)) #true]
```

```
    [(in x (x_1 ...)) #false])

  ;; ---------------------------------------------------------------
-------------
  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

  (module+ test
    (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
    (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
    (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

  (define-metafunction Lambda
    =α : any any -> boolean
    [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

  ;; a Racket definition for use in Racket positions
  (define (=α/racket x y) (term (=α ,x ,y)))

  ;; (sd e) computes the static distance version of e
  (define-extended-language SD Lambda
    (e ::= .... (K n))
    (n ::= natural))

  (define SD? (redex-match? SD e))

  (module+ test
    (define sd1 (term (K 1)))
    (define sd2 (term 1))

    (test-equal (SD? sd1) #true))

  (define-metafunction SD
    sd : any -> any
    [(sd any_1) (sd/a any_1 ())])

  (module+ test
    (test-equal (term (sd/a x ())) (term x))
    (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
    (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
                (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
    (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
                (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
    (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
                (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

  (define-metafunction SD
```

```
  sd/a : any ((x ...) ...) -> any
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ;; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
  [(sd/a (lambda (x ...) any_1) (any_rest ...))
   (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
  [(sd/a (any_fun any_arg ...) (any_rib ...))
   ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
  [(sd/a any_1 any)
   ;; free variable, constant, etc
   any_1])


;; ---------------------------------------------------------------------
-------------
;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)

(module+ test
  (test-equal (term (subst ([1 x][2 y]) x)) 1)
  (test-equal (term (subst ([1 x][2 y]) y)) 2)
  (test-equal (term (subst ([1 x][2 y]) z)) (term z))
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
              (term (lambda (z w) (1 2))))
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
              (term (lambda (z w) (lambda (x) (x 2))))
              #:equiv =α/racket)
  (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
              (term ((lambda (x) (1 x)) 2))
              #:equiv =α/racket)
  (test-equal (term (subst (((lambda (x) y) x)) (lambda (y) x)))
              (term (lambda (y1) (lambda (x) y)))
              #:equiv =α/racket))

(define-metafunction Lambda
  subst : ((any x) ...) any -> any
  [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
  [(subst [(any_1 x_1) ... ] x) x]
  [(subst [(any_1 x_1) ... ] (lambda (x ...) any_body))
   (lambda (x_new ...)
     (subst ((any_1 x_1) ...)
            (subst-raw ((x_new x) ...) any_body)))
   (where  (x_new ...)  ,(variables-not-in (term (any_body any_1 ...)) (term (x ...))))]
  [(subst [(any_1 x_1) ... ] (any ...)) ((subst [(any_1 x_1) ... ] any) ...)]
```

```
    [(subst [(any_1 x_1) ... ] any_*) any_*])

  (define-metafunction Lambda
    subst-raw : ((x x) ...) any -> any
    [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
    [(subst-raw ((x_n1 x_o1) ... ) x) x]
    [(subst-raw ((x_n1 x_o1) ... ) (lambda (x ...) any))
     (lambda (x ...) (subst-raw ((x_n1 x_o1) ... ) any))]
    [(subst-raw [(any_1 x_1) ... ] (any ...))
     ((subst-raw [(any_1 x_1) ... ] any) ...)]
    [(subst-raw [(any_1 x_1) ... ] any_*) any_*])

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-results))
```

## 2.14 "close.rkt"

```
  #lang racket

  ;; a function that can close over the free variables of an expression

  (provide
   ;; RACKET
   ;; [Any-> Boolean: valid expression] ->
   ;;       [Lambda.e #:init [i \x.x] -> Lambda.e]
   ;; ((close-over-fv-with lambda?) e) closes over all free variables in
   ;; a Lambda term (or sublanguage w/ no new binding constructs) by
   ;; binding them to (term (lambda (x) x))
   ;; ((close-over-fv-with lambda?) e #:init i)
   ;;       like above but binds free vars to i
   close-over-fv-with
   ;; any -> (x ...)
   ;; computes free variables of given term
   fv)

  (require redex "common.rkt")

  ;; -----------------------------------------------------
  (module+ test
    ;; show two dozen terms
    (redex-check Lambda e #;displayln (term e)
```

```
                #:attempts 12
                #:prepare (close-over-fv-with lambda?))
    ;; see 0, can't work
    #;
    (redex-check Lambda e #;displayln (term e)
                #:attempts 12
                #:prepare (λ (x) ((close-over-fv-
with lambda?) x #:init 0))))

  (define ((close-over-fv-with lambda?) e #:init (i (term (lambda (x) x))))
    ;; this is to work around a bug in redex-check; doesn't always work
    (if (lambda? e) (term (close ,e ,i)) i))

  (define-metafunction Lambda
    close : any any -> any
    [(close any_1 any_2)
     (let ([x any_2] ...) any_1)
     (where (x ...) (unique (fv any_1)))])

  (define-metafunction Lambda
    ;;  let : ((x e) ...) e -> e but e plus hole
    let : ((x any) ...) any -> any
    [(let ([x_lhs any_rhs] ...) any_body)
     ((lambda (x_lhs ...) any_body) any_rhs ...)])

  (define-metafunction Lambda
    unique : (x ...) -> (x ...)
    [(unique ()) ()]
    [(unique (x_1 x_2 ...))
     (unique (x_2 ...))
     (where #true (in x_1 (x_2 ...)))]
    [(unique (x_1 x_2 ...))
     (x_1 x_3 ...)
     (where (x_3 ...) (unique (x_2 ...)))])

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-equal (term (fv x)) (term (x)))
    (test-equal (term (fv (lambda (x) x))) (term ()))
    (test-equal (term (fv (lambda (x) (y z x)))) (term (y z))))

  (define-metafunction Lambda
    fv : any -> (x ...)
    [(fv x) (x)]
    [(fv (lambda (x ...) any_body))
```

```
     (subtract (x_e ...) x ...)
     (where (x_e ...) (fv any_body))]
    [(fv (any_f any_a ...))
     (x_f ... x_a ... ...)
     (where (x_f ...) (fv any_f))
     (where ((x_a ...) ...) ((fv any_a) ...))]
    [(fv any) ()])

  ;; ---------------------------------------------------------------
-------------
  ;; (subtract (x ...) x_1 ...) removes x_1 ... from (x ...)

  (module+ test
    (test-equal (term (subtract (x y z x) x z)) (term (y))))

  (define-metafunction Lambda
    subtract : (x ...) x ... -> (x ...)
    [(subtract (x ...)) (x ...)]
    [(subtract (x ...) x_1 x_2 ...)
     (subtract (subtract1 (x ...) x_1) x_2 ...)])

  (module+ test
    (test-equal (term (subtract1 (x y z x) x)) (term (y z))))

  (define-metafunction Lambda
    subtract1 : (x ...) x -> (x ...)
    [(subtract1 (x_1 ... x x_2 ...) x)
     (x_1 ... x_2new ...)
     (where (x_2new ...) (subtract1 (x_2 ...) x))
     (where #false (in x (x_1 ...)))]
    [(subtract1 (x ...) x_1) (x ...)])
```

## 2.15 "tc-common.rkt"

```
  #lang racket

  (provide
   ;; language
   TLambda-tc

   ;; (extend Γ (x t) ...) add (x t) to Γ so that x is found before other x-
s
   extend
```

```
   ;; (lookup Γ x) retrieves x's type from Γ
   lookup)

  (require redex)

  ;; ------------------------------------------------------------------
-------------
  (define-language TLambda-tc
    (e ::= n + x (lambda ((x_!_ t) ...) e) (e e ...))
    (n ::= natural)
    (t ::= int (t ... -> t))
    (Γ ::= ((x t) ...))
    (x ::= variable-not-otherwise-mentioned))

  (define tlambda? (redex-match? TLambda-tc e))

  ;; ------------------------------------------------------------------
-------------
  ;; (extend Γ (x t) ...) add (x t) to Γ so that x is found before other x-
s

  (module+ test
    (test-equal (term (extend () (x int))) (term ((x int)))))

  (define-metafunction TLambda-tc
    extend : Γ (x any) ... -> any
    [(extend ((x_Γ any_Γ) ...) (x any) ...) ((x any) ...(x_Γ any_Γ) ...)])

  ;; ------------------------------------------------------------------
-------------
  ;; (lookup Γ x) retrieves x's type from Γ

  (module+ test
    (test-equal (term (lookup ((x int) (x (int -> int)) (y int)) x)) (term int))
    (test-equal (term (lookup ((x int) (x (int -> int)) (y int)) y)) (term int)))

  (define-metafunction TLambda-tc
    lookup : any x -> any or #f
    [(lookup ((x_1 any_1) ... (x any_t) (x_2 any_2) ...) x)
     any_t
     (side-condition (not (member (term x) (term (x_1 ...)))))]
    [(lookup any_1 any_2)
     #f])
```

## 2.16 "extend-lookup.rkt"

```
#lang racket

(provide
 ;; (extend σ (x ...) (v ...)) adds (x v) ... to σ
 extend

 ;; (lookup σ x) retrieves x's value from σ
 lookup)


;; ---------------------------------------------------------------
--
(require redex "common.rkt")

(define-metafunction Lambda
  extend : ((x any) ...)  (x ...) (any ...) -> ((x any) ...)
  [(extend ((x any) ...) (x_1 ...) (any_1 ...))
   ((x_1 any_1) ... (x any) ...)])

(define-metafunction Lambda
  lookup : ((x any) ...) x -> any or #f
  [(lookup ((x_1 any_1) ... (x any_t) (x_2 any_2) ...) x)
   any_t
   (side-condition (not (member (term x) (term (x_1 ...)))))]
  [(lookup any_1 any_2)
   #f])
```

# 3 Extended Exercises

This section offers some Redex challenges of varying complexity. They are broken up into separate sections, alternating between problems and sample solutions.

# Contents

## 3.1 Problem: Objects

Design a small model of untyped objects: a language, scoping, an adapted substitution function, and a (standard or regular) reduction system.

Start with the simplification of objects as multi-entry functions:

```
(define-language Object
  (e ::=
     n
     y
     (object (m (x) e) ...)
     (send e m e))
  (y ::=
     x
     this)
  (n ::= number)
  (m ::= variable-not-otherwise-mentioned)
  (x ::= variable-not-otherwise-mentioned))
```

Sometimes we wish to treat `this` like a variable and at other times, we want to exclude it from this world. To support this two-faced treatment, the grammar includes the syntactic category of *y*, which consists of the set *x* of variables and `this`.

## 3.2 Solution: Objects

This solution shows how numbers are interpreted as objects and messages to these numbers might include symbols such as `+`. Consider extending this solution with some of the following:

- recognize stuck states for expressions such as `(send n + (object ...))`, `(send (object ...) + n)`, or `(send o m v)` where *o* does not have an entry point labeled *m*.
- a `clone` operation for objects
- an `update` operation for objects that adds a new method
- and the inclusion of fields.

```
#lang racket

;; a model of simple object programming (no updater, no prototype, no clone)
```

```
  (require redex (only-in "common.rkt" in))

  ;; ---------------------------------------------------------------
-------------
  ;; syntax

  (define-language Object
    (e ::= n y (object (m (x) e) ...) (send e m e))
    (y ::= x this)
    (n ::= natural)
    (m ::= variable-not-otherwise-mentioned)
    (x ::= variable-not-otherwise-mentioned))

  ;; ---------------------------------------------------------------
-------------
  ;; examples

  (define help
    (term (object [help (x) x])))

  (define p-good
    (term
     (send
      (object [get(x) this]
              [set(x) x])
      set
      ,help)))

  (define p-8
    (term
     (send (object [get(x) this] [set(x) (send x + 3)]) set 5)))

  (module+ test
    (test-equal (redex-match? Object e help) #true)
    (test-equal (redex-match? Object e p-good) #true))

  ;; ---------------------------------------------------------------
-------------
  ;; scope

  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent
  (module+ test
    (test-equal (term (=α (object (help (x) x)) (object (help (y) y)) )) #true)
    (test-equal (term (=α (object (help (x) x)) (object (main (y) y)) )) #false))
```

```
(define-metafunction Object
  =α : any any -> boolean
  [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

;; a Racket definition for use in Racket positions
(define (=α/racket x y) (term (=α ,x ,y)))

;; (sd e) computes the static distance version of e
(define-extended-language SD Object
  (e ::= .... (K n))
  (n ::= natural))

(define SD? (redex-match? SD e))

(module+ test
  (define sd1 (term (K 1)))
  (define sd2 (term 1))

  (test-equal (SD? sd1) #true))

(define-metafunction SD
  sd : any -> any
  [(sd any_1) (sd/a any_1 ())])

(module+ test
  (define help-sd
    (term (object [help () (K 0)])))
  (define p-good-sd
    (term
     (send
      (object [get(x) this]
              [set(x) (K 0)])
      set
      ,help-sd)))
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x (y z x))) (term (K 2)))
  (test-equal (term (sd ,help)) help-sd))

(define-metafunction SD
  sd/a : any (x ...) -> any
  ;; bound variable
  [(sd/a x (x_1 ... x x_2 ...))
   (K n_rib)
   (where n_rib ,(length (term (x_1 ...))))
   (where #false (in x (x_1 ...)))]
  ;; free variable
```

```
    [(sd/a x (x_1 ...)) x]
    [(sd/a (object (m (x) any_1) ...) (any_rest ...))
     (object (m () (sd/a any_1 (x any_rest ...))) ...)]
    [(sd/a (send any_fun m any_arg) (any_rib ...))
     (send (sd/a any_fun (any_rib ...)) m (sd/a any_arg (any_rib ...)))]
    [(sd/a any (x_1 ...)) any])

  ;; ---------------------------------------------------------------
-------------
  ;; substitution

  ;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)
  (module+ test
    (test-equal (term (subst ([1 x][2 y]) x)) 1)
    (test-equal (term (subst ([1 x][2 y]) y)) 2)
    (test-equal (term (subst ([1 x][2 y]) z)) (term z))
    (test-equal
     (term (subst ([1 x][2 y]) (object (m (z) x))))
     (term (object (m (z) 1)))
     #:equiv =α/racket)
    (test-equal
     (term (subst ([1 x][2 y]) (object (m (z) (object (n (x) x))))))
     (term (object (m (z) (object (n (x) x)))))
     #:equiv =α/racket)
    (test-equal
     (term (subst ([1 x][2 y]) (object (m (z) (object (n (y) x))))))
     (term (object (m (z) (object (n (y) 1)))))
     #:equiv =α/racket)
    (test-equal
     (term (subst ([(object (x) y) x][2 y]) (object (m (z) (object (n (y) x))))))
     (term (object (m (z) (object (n (y1) (object (x) y))))))
     #:equiv =α/racket)

    (test-equal
     (term
      (subst ([(object (help (x) x)) x]
              [(object (get (x) this) (set (x) x)) this])
             x))
     help
     #:equiv =α/racket))

  (define-metafunction Object
    subst : ((any y) ...) any -> any
    [(subst [(any_1 y_1) ... (any_x x) (any_2 y_2) ...] x) any_x]
    [(subst [(any_1 y_1) ... ] x) x]
    [(subst [(any_1 y_1) ... ] (object (m (x) any_m) ...))
```

```
     (object
      (m (y_new) (subst ((any_1 y_1) ...) (subst-
raw ((y_new x) ...) any_m))) ...)
     (where (y_new ...) (fresh-in any_m ... any_1 ... (x ...)))]
    [(subst [(any_1 y_1) ... ] (any ...)) ((subst [(any_1 y_1) ... ] any) ...)]
    [(subst [(any_1 y_1) ... ] any_*) any_*])

  (define-metafunction Object
    subst-raw : ((y y) ...) any -> any
    [(subst-raw ((y_n1 y_o1) ... (y_new x) (y_n2 y_o2) ...) x) y_new]
    [(subst-raw ((y_n1 y_o1) ... ) x) x]
    [(subst-raw ((y_n1 y_o1) ... ) (object (m (x) any_m) ...))
     (object (m (x) (subst-raw ((y_n1 y_o1) ... ) any_m)) ...)]
    [(subst-raw [(any_1 y_1) ... ] (any ...))
     ((subst-raw [(any_1 y_1) ... ] any) ...)]
    [(subst-raw [(any_1 y_1) ... ] any_*) any_*])

  ;; (fresh-in any ... (x ...)) generates a sequence of variables
  ;; like x ... not in any ...
  (define-metafunction Object
    fresh-in : any ... (x ...) -> (x ...)
    [(fresh-in any ... (x ...))
     ,(variables-not-in (term (any ...)) (term (x ...)))])

  ;; ---------------------------------------------------------------------
-------------
  ;; the object calculus (standard reduction)

  (define-extended-language Object-calculus Object
    (v ::= n (object (m (x) e) ...))
    (E ::= hole (send E m e) (send v m E)))

  (module+ test
    #;
    (traces -->obj p-good)
    (test-->> -->obj #:equiv =α/racket p-good help)
    (test-->> -->obj #:equiv =α/racket p-8 8))

  (define -->obj
    (reduction-relation
     Object-calculus
     (--> (in-hole E (send (name THIS
                                 (object (m_left (x_left) e_left) ...
                                         (m (x) e)
                                         (m_right (x_right) e_right) ...))
                           m
```

```
                          v))
          (in-hole E (subst ([v x][THIS this]) e))
          send)
     (--> (in-hole E (send n_1 + n_2))
          (in-hole E ,(+ (term n_1) (term n_2)))
          add)))

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-results))



  ;;; ---------------------------------------------------------------
  ;;; common.rkt starts here

  #lang racket

  ;; basic definitions for the Redex Summer School 2015

  (provide
   ;; Language
   Lambda

   ;; Any -> Boolean
   ;; is the given value in the expression language?
   lambda?

   ;; x (x ...) -> Boolean
   ;; (in x (x_1 ...)) determines whether x occurs in x_1 ...
   in

   ;; Any Any -> Boolean
   ;; (=α/racket e_1 e_2) determines whether e_1 is α-
equivalent to e_2
   ;; e_1, e_2 are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   =α/racket

   ;; ((Lambda x) ...) Lambda -> Lambda
   ;; (subs ((e_1 x_1) ...) e) substitures e_1 for x_1 ... in e
   ;; e_1, ... e are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   subst)
```

```
  ;; ---------------------------------------------------------------
-------------
  (require redex)

  (define-language Lambda
    (e ::=
       x
       (lambda (x_!_ ...) e)
       (e e ...))
    (x ::= variable-not-otherwise-mentioned))

  (define lambda? (redex-match? Lambda e))

  (module+ test
    (define e1 (term y))
    (define e2 (term (lambda (y) y)))
    (define e3 (term (lambda (x y) y)))
    (define e4 (term (,e2 e3)))

    (test-equal (lambda? e1) #true)
    (test-equal (lambda? e2) #true)
    (test-equal (lambda? e3) #true)
    (test-equal (lambda? e4) #true)

    (define eb1 (term (lambda (x x) y)))
    (define eb2 (term (lambda (x y) 3)))

    (test-equal (lambda? eb1) #false)
    (test-equal (lambda? eb2) #false))

  ;; ---------------------------------------------------------------
-------------
  ;; (in x x_1 ...) is x a member of (x_1 ...)?

  (module+ test
    (test-equal (term (in x (y z x y z))) #true)
    (test-equal (term (in x ())) #false)
    (test-equal (term (in x (y z w))) #false))

  (define-metafunction Lambda
    in : x (x ...) -> boolean
    [(in x (x_1 ... x x_2 ...)) #true]
    [(in x (x_1 ...)) #false])

  ;; ---------------------------------------------------------------
```

```
-------------
  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

  (module+ test
    (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
    (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
    (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

  (define-metafunction Lambda
    =α : any any -> boolean
    [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

  ;; a Racket definition for use in Racket positions
  (define (=α/racket x y) (term (=α ,x ,y)))

  ;; (sd e) computes the static distance version of e
  (define-extended-language SD Lambda
    (e ::= .... (K n))
    (n ::= natural))

  (define SD? (redex-match? SD e))

  (module+ test
    (define sd1 (term (K 1)))
    (define sd2 (term 1))

    (test-equal (SD? sd1) #true))

  (define-metafunction SD
    sd : any -> any
    [(sd any_1) (sd/a any_1 ())])

  (module+ test
    (test-equal (term (sd/a x ())) (term x))
    (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
    (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
                (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
    (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
                (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
    (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
                (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

  (define-metafunction SD
    sd/a : any ((x ...) ...) -> any
    [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
     ;; bound variable
```

```
    (K n_rib n_pos)
    (where n_rib ,(length (term ((x_1 ...) ...))))
    (where n_pos ,(length (term (x_0 ...))))
    (where #false (in x (x_1 ... ...)))]
   [(sd/a (lambda (x ...) any_1) (any_rest ...))
    (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
   [(sd/a (any_fun any_arg ...) (any_rib ...))
    ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
   [(sd/a any_1 any)
    ;; free variable, constant, etc
    any_1])


  ;; ---------------------------------------------------------------
-------------
  ;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)

  (module+ test
    (test-equal (term (subst ([1 x][2 y]) x)) 1)
    (test-equal (term (subst ([1 x][2 y]) y)) 2)
    (test-equal (term (subst ([1 x][2 y]) z)) (term z))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
                (term (lambda (z w) (1 2))))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
                (term (lambda (z w) (lambda (x) (x 2))))
                #:equiv =α/racket)
    (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
                (term ((lambda (x) (1 x)) 2))
                #:equiv =α/racket)
    (test-equal (term (subst (((lambda (x) y) x)) (lambda (y) x)))
                (term (lambda (y1) (lambda (x) y)))
                #:equiv =α/racket))

  (define-metafunction Lambda
    subst : ((any x) ...) any -> any
    [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
    [(subst [(any_1 x_1) ... ] x) x]
    [(subst [(any_1 x_1) ... ] (lambda (x ...) any_body))
     (lambda (x_new ...)
       (subst ((any_1 x_1) ...)
              (subst-raw ((x_new x) ...) any_body)))
     (where  (x_new ...)  ,(variables-not-in (term (any_body any_1 ...)) (term (x ...))))]
    [(subst [(any_1 x_1) ... ] (any ...)) ((subst [(any_1 x_1) ... ] any) ...)]
    [(subst [(any_1 x_1) ... ] any_*) any_*])

  (define-metafunction Lambda
```

```
    subst-raw : ((x x) ...) any -> any
    [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
    [(subst-raw ((x_n1 x_o1) ... ) x) x]
    [(subst-raw ((x_n1 x_o1) ... ) (lambda (x ...) any))
     (lambda (x ...) (subst-raw ((x_n1 x_o1) ... ) any))]
    [(subst-raw [(any_1 x_1) ... ] (any ...))
     ((subst-raw [(any_1 x_1) ... ] any) ...)]
    [(subst-raw [(any_1 x_1) ... ] any_*) any_*])

  ;; ---------------------------------------------------------------------
-------------
  (module+ test
    (test-results))
```

## 3.3 Problem: Types

Design a reduction system that models type checking by reducing terms to their types. Formulate a metafunction that maps terms to types or `#false`:

```
(define-metafunction TLambda-tc
  tc : e -> t or #false
  ; more here
  .....)
```

It produces `#false` when type checking fails.

You start with the normal term language:

```
(define-language TLambda
  (e ::= n x (lambda (x t) e) (e e) (+ e e))
  (n ::= number)
  (t ::= int (t -> t))
  (x ::= variable-not-otherwise-mentioned))
```

The reduction systems gradually reduces terms to types, like this:

```
(define type-checking
  (reduction-relation
    (--> (in-hole C n) (in-hole C int))
    (--> (in-hole C (+ int int)) (in-hole C int))
    ; you need more here
    .....))
```

**Hint** The key is to design a language of contexts that extends the given expression language so the reduction system can find all possible terms.

Consider adding `if` and `let` expressions to the language once you have the core model working.

## 3.4 Solution: Types

Also consult chapter III.23 in the Redex book for ideas.

```
#lang racket

;; a type reduction-based approach to type checking

(require redex "common.rkt")

;; ---------------------------------------------------------------
-------------
;; syntax
(define-language TLambda
  (e ::= n x (lambda (x t) e) (e e) (+ e e))
  (n ::= number)
  (t ::= int (t -> t))
  (x ::= variable-not-otherwise-mentioned))

(define in-TLambda? (redex-match? TLambda e))

;; ---------------------------------------------------------------
-------------
;; examples

(define e1
  (term (lambda (x int) (lambda (f (int -> int)) (+ (f (f x)) (f x))))))

(define e2
  (term
   (lambda (x int)
     (lambda (f ((int -> int) -> int))
       (f x)))))
(define e3 (term (lambda ((x int)) (int -> int))))

(module+ test
  (test-equal (in-TLambda? e1) #true)
  (test-equal (in-TLambda? e2) #true)
```

```
    (test-equal (in-TLambda? e3) #false))

  ;; ---------------------------------------------------------------
-------------
  ;; (⊢ Γ e t) -- the usual type judgment for an LC language

  (define-extended-language TLambda-tc TLambda
    (e ::= .... t (t -> e))
    (C ::= hole (lambda (x t) C) (C e) (e C) (+ C e) (+ e C) (t -> C)))

  (module+ test
    (test-equal (term (tc ,e1)) (term (int -> ((int -> int) -> int))))

    ;; a failure -- no types are returned
    (test-equal (term (tc ,e2)) #false))

  (define-metafunction TLambda-tc
    tc : e -> t or #false
    [(tc t) t]
    [(tc e)
     (tc e_again)
     (where (e_again e_more ...) ,(apply-reduction-
relation ->tc (term e)))]
    [(tc e_stuck) #false])

  (define ->tc
    (reduction-relation
     TLambda-tc
     (--> (in-hole C n) (in-hole C int))
     (--> (in-hole C (+ int int)) (in-hole C int))
     (--> (in-hole C (lambda (x t) e)) (in-hole C (t -> (subst ((t x)) e))))
     (--> (in-hole C ((t -> t_range) t)) (in-hole C t_range))))

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-results))



  ;;; ---------------------------------------------------------------
  ;;; common.rkt starts here

  #lang racket

  ;; basic definitions for the Redex Summer School 2015
```

```
  (provide
   ;; Language
   Lambda

   ;; Any -> Boolean
   ;; is the given value in the expression language?
   lambda?

   ;; x (x ...) -> Boolean
   ;; (in x (x_1 ...)) determines whether x occurs in x_1 ...
   in

   ;; Any Any -> Boolean
   ;; (=α/racket e_1 e_2) determines whether e_1 is α-
equivalent to e_2
   ;; e_1, e_2 are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   =α/racket

   ;; ((Lambda x) ...) Lambda -> Lambda
   ;; (subs ((e_1 x_1) ...) e) substitures e_1 for x_1 ... in e
   ;; e_1, ... e are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   subst)

  ;; ---------------------------------------------------------------
-------------
  (require redex)

  (define-language Lambda
    (e ::=
       x
       (lambda (x_!_ ...) e)
       (e e ...))
    (x ::= variable-not-otherwise-mentioned))

  (define lambda? (redex-match? Lambda e))

  (module+ test
    (define e1 (term y))
    (define e2 (term (lambda (y) y)))
    (define e3 (term (lambda (x y) y)))
    (define e4 (term (,e2 e3)))

    (test-equal (lambda? e1) #true)
```

```
    (test-equal (lambda? e2) #true)
    (test-equal (lambda? e3) #true)
    (test-equal (lambda? e4) #true)

    (define eb1 (term (lambda (x x) y)))
    (define eb2 (term (lambda (x y) 3)))

    (test-equal (lambda? eb1) #false)
    (test-equal (lambda? eb2) #false))

  ;; ---------------------------------------------------------------
-------------
  ;; (in x x_1 ...) is x a member of (x_1 ...)?

  (module+ test
    (test-equal (term (in x (y z x y z))) #true)
    (test-equal (term (in x ())) #false)
    (test-equal (term (in x (y z w))) #false))

  (define-metafunction Lambda
    in : x (x ...) -> boolean
    [(in x (x_1 ... x x_2 ...)) #true]
    [(in x (x_1 ...)) #false])

  ;; ---------------------------------------------------------------
-------------
  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

  (module+ test
    (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
    (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
    (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

  (define-metafunction Lambda
    =α : any any -> boolean
    [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

  ;; a Racket definition for use in Racket positions
  (define (=α/racket x y) (term (=α ,x ,y)))

  ;; (sd e) computes the static distance version of e
  (define-extended-language SD Lambda
    (e ::= .... (K n))
    (n ::= natural))

  (define SD? (redex-match? SD e))
```

```
(module+ test
  (define sd1 (term (K 1)))
  (define sd2 (term 1))

  (test-equal (SD? sd1) #true))

(define-metafunction SD
  sd : any -> any
  [(sd any_1) (sd/a any_1 ())])

(module+ test
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
  (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
              (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
  (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
              (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
  (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
              (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

(define-metafunction SD
  sd/a : any ((x ...) ...) -> any
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ;; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
  [(sd/a (lambda (x ...) any_1) (any_rest ...))
   (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
  [(sd/a (any_fun any_arg ...) (any_rib ...))
   ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
  [(sd/a any_1 any)
   ;; free variable, constant, etc
   any_1])


;; ---------------------------------------------------------------
-------------
;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)

(module+ test
  (test-equal (term (subst ([1 x][2 y]) x)) 1)
  (test-equal (term (subst ([1 x][2 y]) y)) 2)
  (test-equal (term (subst ([1 x][2 y]) z)) (term z))
```

```
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
              (term (lambda (z w) (1 2))))
  (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
              (term (lambda (z w) (lambda (x) (x 2))))
              #:equiv =α/racket)
  (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
              (term ((lambda (x) (1 x)) 2))
              #:equiv =α/racket)
  (test-equal (term (subst (((lambda (x) y) x)) (lambda (y) x)))
              (term (lambda (y1) (lambda (x) y)))
              #:equiv =α/racket))

(define-metafunction Lambda
  subst : ((any x) ...) any -> any
  [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
  [(subst [(any_1 x_1) ... ] x) x]
  [(subst [(any_1 x_1) ... ] (lambda (x ...) any_body))
   (lambda (x_new ...)
     (subst ((any_1 x_1) ...)
            (subst-raw ((x_new x) ...) any_body)))
   (where  (x_new ...)  ,(variables-not-in (term (any_body any_1 ...)) (term (x ...))))]
  [(subst [(any_1 x_1) ... ] (any ...)) ((subst [(any_1 x_1) ... ] any) ...)]
  [(subst [(any_1 x_1) ... ] any_*) any_*])

(define-metafunction Lambda
  subst-raw : ((x x) ...) any -> any
  [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
  [(subst-raw ((x_n1 x_o1) ... ) x) x]
  [(subst-raw ((x_n1 x_o1) ... ) (lambda (x ...) any))
   (lambda (x ...) (subst-raw ((x_n1 x_o1) ... ) any))]
  [(subst-raw [(any_1 x_1) ... ] (any ...))
   ((subst-raw [(any_1 x_1) ... ] any) ...)]
  [(subst-raw [(any_1 x_1) ... ] any_*) any_*])

;; ---------------------------------------------------------------------
-------------
(module+ test
  (test-results))
```

## 3.5 Problem: Missionaries and Cannibals

Here is a old puzzle from the 1800s:

> "Once upon a time, three cannibals were guiding three missionaries through a jungle. They were on their way to the nearest mission station. After some time, they arrived at a wide river, filled with deadly snakes and fish. There was no way to cross the river without a boat. Fortunately, they found a rowing boat with two oars after a short search. Unfortunately, the boat was too small to carry all of them. It could barely carry two people at a time. Worse, because of the river's width someone had to row the boat back.
>
> "Since the missionaries could not trust the cannibals, they had to figure out a plan to get all six of them safely across the river. The problem was that these cannibals would kill and eat missionaries as soon as there were more cannibals than missionaries at some place. Thus our missionaries had to devise a plan that guaranteed that there were never any missionaries in the minority at either side of the river. The cannibals, however, could be trusted to cooperate otherwise. Specifically, they wouldn't abandon any potential food, just as the missionaries wouldn't abandon any potential converts."

Formulate a reduction system that solves this puzzle. Use `traces` to visualize the solution space. You want to impose a side condition on the rules so that no boat is sent back to the other side of the river when a configuration represents a solutions. The other side conditions just ensure that no missionary will be eaten.

Consider running `traces` with a subject-reduction test. The predicate you want to check is that the parties on both sides of the river are safe.

## 3.6 Solution: Missionaries and Cannibals

```
#lang racket

;; solving the missionaries-and-cannibals problem with Redex

(require redex)

;; ------------------------------------------------------------------
-------------
;; the problem space syntax

(define-language MC
  (configuration ::= (population boat population))
  (population ::= (mc ...))
  (boat ::= L R)
  (mc ::= c m))

;; ------------------------------------------------------------------
-------------
```

```
  ;; constraints

  (define-metafunction MC
    ok : population -> boolean
    [(ok (mc ...))
     ,(let ((m (for/sum ((mc (term (mc ...))) #:when (eq? 'm mc)) 1))
            (c (for/sum ((mc (term (mc ...))) #:when (eq? 'c mc)) 1)))
        (or (zero? m) (>= m c)))])

  ;; a subject reduction test (which sadly failed for the first draft)
  (define-metafunction MC
    ok-state : configuration -> boolean
    [(ok-state ((mc_l ...) any (mc_r ...)))
     ,(and (term (ok (mc_l ...))) (term (ok (mc_r ...))))])

  ;; ---------------------------------------------------------------------
-------------
  ;; a reduction relation that searches the state space

  (define mc-->
    (reduction-relation
     MC
     (--> [(mc_l1 ... mc_* mc_l2 ... mc_+ mc_l3 ...) L (mc_r ...)]
          ;; move two people from left to right
          [(mc_l1 ... mc_l2 ... mc_l3 ...) R (mc_* mc_+ mc_r ...)]
          (where population_left (mc_l1 ... mc_l2 ... mc_l3 ...))
          (where population_right (mc_* mc_+ mc_r ...))
          (where #true (ok population_left))
          (where #true (ok population_right))
          move-2-left-to-right)
     (--> [(mc mc_1 ...) R (mc_r1 ... mc_* mc_r2 ...)]
          ;; move one person from right to left
          [(mc_* mc mc_1 ...) L (mc_r1 ... mc_r2 ...)]
          (where population_left (mc_* mc mc_1 ...))
          (where population_right (mc_r1 ... mc_r2 ...))
          (where #true (ok population_left))
          (where #true (ok population_right))
          move-1-right-to-left)
     (--> [(mc mc_1 ...) R (mc_r1 ... mc_* mc_r2 ... mc_+ mc_r3 ...)]
          ;; move two people from right to left
          [(mc_* mc_+ mc mc_1 ...) L (mc_r1 ... mc_r2 ... mc_r3 ...)]
          (where population_left (mc_* mc_+ mc mc_1 ...))
          (where population_right (mc_r1 ... mc_r2 ... mc_r3 ...))
          (where #true (ok population_left))
          (where #true (ok population_right))
          move-2-right-to-left)))
```

```
  ;; ---------------------------------------------------------------
-------------
  (module+ main
    (traces mc--> (term ((m m m c c c) L ()))
            #:pred (lambda (e) (term (ok-state ,e)))))
```

## 3.7 Problem: Towers of Hanoi

Implement a solver for Tower of Hanoi as a reduction relation, where a step by the reduction corresponds to a move in the game. You can implement the solver as an exploration of all possible game moves via the reduction relation, checking whether a solution state is reachable.

**Hints** While you can implement the game using just `define-language` and `reduction-relation`, a metafunction that checks whether a stack of ties will "accept" a given additional tile makes the reduction easier to write. Among the possible choices for representing a tile, a list of •s works well and looks nice.

## 3.8 Solution: Towers of Hanoi

```
  #lang racket

  ;; solving towers of Hanoi by searching the solution space

  (require redex)

  ;; ---------------------------------------------------------------
-------------
  ;; the state space of configurations
  (define-language L
    [chunk *]
    [tile (chunk ...)]
    [stack (side-condition [tile_1 ...]
                           (term (stacked [tile_1 ...])))]
    [state (stack ...)])

  ;; ---------------------------------------------------------------
-------------
  ;; checking the stacks

  (define-metafunction L
```

```
  stacked : [tile ...] -> any
  [(stacked []) #t]
  [(stacked [tile_0 tile_1 ...])
   (stacked [tile_1 ...])
   (judgment-holds (accepts [tile_1 ...] tile_0 ))])

(define-judgment-form L
  #:mode (accepts I I)
  #:contract (accepts stack tile)
  [-----------------
   (accepts [] tile)]
  [-----------------
   (accepts [(chunk_0 ... chunk_1 ..._1) tile ...]
            (chunk_1 ..._1))])

;; ---------------------------------------------------------------
-------------
;; the redution system

(module+ test
  (test-->>∃ -->hanoi
             (term ([(*) (* *) (* * *)] [] []))
             (term ([]                  [] [(*) (* *) (* * *)]))))

(define -->hanoi
  (reduction-relation
   L
   [--> (stack_0 ... [tile_0 tile_1 ...]
         stack_1 ... [tile_2 ...]
         stack_3 ...)
        (stack_0 ... [tile_1 ...]
         stack_1 ... [tile_0 tile_2 ...]
         stack_3 ...)
        (judgment-holds (accepts [tile_2 ...] tile_0))]
   [--> (stack_0 ... [tile_1 ...]
         stack_1 ... [tile_0 tile_2 ...]
         stack_3 ...)
        (stack_0 ... [tile_0 tile_1 ...]
         stack_1 ... [tile_2 ...]
         stack_3 ...)
        (judgment-holds (accepts [tile_1 ...] tile_0))]))

(module+ test
  (test-results))

;; rendering the search
```

```
(module+ main
  (traces -->hanoi (term ([(*) (* *) (* * *)] [] []))))
```

## 3.9 Problem: GC

Consider the language of stored binary trees:

```
(define-language L
 [V number
    (cons σ σ)]
 [Σ ([σ V] ...)]
 [σ variable-not-otherwise-mentioned])
```

Design the `-->gc` reduction relation, which implements garbage collection. The `-->gc` reduction relation operates on a configuration that combines the store Σ, a set of "gray" addresses, i.e., σs) to be explored (the addresses of the initially reachable objects, also called roots, plus a set of "black" addresses (initially empty). Each step operates on one gray address, adjusting the gray and black sets based on the address's value in the store. No more steps are possible when the set of gray addresses goes empty, at which point every address not in the black list can be pruned from the store.

## 3.10 Solution: GC

```
#lang racket

;; a model of garbage collection for binary trees in a store

(require redex)

;; ------------------------------------------------------------------
-------------
;; syntax
(define-language L
  [V number
     (cons σ σ)]
  [σ variable-not-otherwise-mentioned]
  [Σ ([σ V] ...)]
  [σs (σ ...)])

;; ------------------------------------------------------------------
-------------
```

```
  ;; set constraints
  (define-judgment-form L
    #:mode (∈ I I)
    #:contract (∈ any (any ...))
    [-----------------
     (∈ any_1 (_ ... any_1 _ ...))])

  (define-judgment-form L
    #:mode (∉ I I)
    #:contract (∉ any (any ...))
    [-----------------
     (∉ any_!_ (any_!_ ...))])

  ;; -----------------------------------------------------------------
-------------
  ;; the reduction system

  (module+ test
    (test-->> -->gc
              (term [([a 1] [b (cons a b)] [c (cons c c)]) (a) ()])
              (term [([a 1] [b (cons a b)] [c (cons c c)]) () (a)]))
    (test-->> -->gc
              (term [([a 1] [b (cons a b)] [c (cons c c)]) (b) ()])
              (term [([a 1] [b (cons a b)] [c (cons c c)]) () (b a)]))
    (test-->> -->gc
              (term [([a 1] [b (cons a b)] [c (cons c c)]) (c) ()])
              (term [([a 1] [b (cons a b)] [c (cons c c)]) () (c)])))

  (define -->gc
    (reduction-relation
     L
     #:domain [Σ σs σs]
     (--> [Σ (σ_g σ_g2 ...) σs_b]
          [Σ (σ_g2 ...) σs_b]
          (judgment-holds (∈ σ_g σs_b))
          "already black")

     (--> [Σ (σ_g σ_g2 ...) (name σs_b (σ_b ...))]
          [Σ (σ_g2 ...) (σ_b ... σ_g)]
          (where (_ ... [σ_g number_g] _ ...) Σ)
          (judgment-holds (∉ σ_g σs_b))
          "number cell")

     (--> [Σ (σ_g σ_g2 ...) (name σs_b (σ_b ...))]
          [Σ (σ_ga σ_gd σ_g2 ...) (σ_b ... σ_g)]
          (where (_ ... [σ_g (cons σ_ga σ_gd)] _ ...) Σ)
```

```
          (judgment-holds (∉ σ_g σs_b))
          "pair cell")))


  (module+ test
    (test-results))
```

## 3.11 Problem: Finite State Machines

Implement a reduction relation that executes a given finite-state machine with a given initial state on a given input sequence. Your representation for finite-state machine should accommodate a non-deterministic set of transition rules.

**Challenge** Implement a metafunction that converts a non-deterministic machine to a deterministic one.

## 3.12 Solution: Finite State Machines

```
  #lang racket

  ;; modeling the transitions in non-deterministic finite-
state machines

  (require redex)

  ;; ---------------------------------------------------------------
-------------
  ;; syntax

  (define-language L
    [FSM (rule ...)]
    [rule [state -> input -> state]]
    [state variable-not-otherwise-mentioned]
    [input variable-not-otherwise-mentioned])

  ;; ---------------------------------------------------------------
-------------
  ;; the reduction system

  (module+ test
    (define fsm1 (term ([a -> x -> b]
                        [a -> y -> c]
```

```
                    [b -> x -> a])))

  (test-->> -->fsm
            (term [,fsm1
                   a
                   (x x y)])
            (term [,fsm1
                   c
                   ()]))
  (test-->> -->fsm
            (term [,fsm1
                   a
                   (x x y x)])
            (term [,fsm1
                   c
                   (x)]))

  (define fsm2 (term ([a -> x -> b]
                      [a -> y -> c]
                      [a -> y -> d]
                      [b -> x -> a]
                      [d -> x -> b])))
  (test-->>∃ -->fsm
             (term [,fsm2
                    a
                    (x x y x)])
             (term [,fsm2
                    b
                    ()])))

(define -->fsm
  (reduction-relation
   L
   #:domain [FSM state (input ...)]
   (--> [FSM state_1 (input_0 input_1 ...)]
        [FSM state_2 (input_1 ...)]
        (where (_ ... [state_1 -> input_0 -> state_2] _ ...)
               FSM))))

(module+ test
  (test-results))
```

## 3.13 Problem: Threads

Add process forking and channel-based communication to a call-by-value functional language. Here is the proposed grammar:

```
(define-language Lambda
  (e ::=
     x (lambda (x_!_ ...) e) (e e ...)
     n (+ e e)
     (if0 e e e)
     (spawn e)
     (put c e)
     (get c)
     (void))
  (n ::= number)
  (c ::= variable-not-otherwise-mentioned)
  (x ::= variable-not-otherwise-mentioned))
```

A `(spawn e)` expression creates a new thread from the given sub-expression, while `put` and `get` expressions allow these threads to communicate. Specifically, when one thread evaluates `(put c v)` and another evaluates `(get c)` for the same `c`, the `get` thread receives value `v` while the `put` thread's expression evaluates to `(void)`.

**Hints** Instead of a single expression, your reductions must deal with a set of expressions, one per thread. Reducing `(spawn e)` in one of these expressions thus adds an `e` to that set; use `(void)` as the result of this action. When any one thread has `(get c)` as its redex and another has `(put c v)`, the two redexes are simultaneously replaced with their contractions.

In Redex, sets are currently realized with sequences. The key difference is that sets are *unordered* and sequences are *ordered*. Keep this in mind when you formulate reduction relations for `put`-`get` communications.

## 3.14 Solution: Threads

```
#lang racket

;; a model of channel-based communication in a by-
value language with threads

(require redex "common.rkt")

(define-language Lambda
  (e ::=
```

```
      x (lambda (x_!_ ...) e) (e e ...)
      n (+ e e)
      (if0 e e e)
      (spawn e)
      (put c e)
      (void)
      (get c))
    (n ::= number)
    (c ::= variable-not-otherwise-mentioned)
    (x ::= variable-not-otherwise-mentioned))

  ;; auxiliary syntax

  ;; a metafunction that acts like a macro in Lambda-calculus
  ;; exercise 3 from Monday afternoon
  (define-metafunction Lambda
    ;;  let : ((x e) ...) e -> e but e plus hole
    let : ((x any) ...) any -> any
    [(let ([x_lhs any_rhs] ...) any_body)
     ((lambda (x_lhs ...) any_body) any_rhs ...)])

  ;; ---------------------------------------------------------------
-------------
  ;; examples

  (define e0 (term (put x 5)))
  (define e1 (term (get x)))
  (define e2 (term (let ([_a (spawn ,e0)] [_b (spawn ,e1)]) 1)))
  (define p0 (term (let ([c y]) ,e2)))

  (module+ test
    (test-equal (redex-match? Lambda e e0) #true)
    (test-equal (redex-match? Lambda e e1) #true)
    (test-equal (redex-match? Lambda e p0) #true))

  ;; ---------------------------------------------------------------
-------------
  ;; a standard reduction relation

  (define-extended-language Lambda-calculus Lambda
    (s ::= (e ...))
    (v ::= n c (void) (lambda (x ...) e))
    (E ::= hole
       (v ... E e ...)
       (+ v ... E e ...)))
```

```
  (define s1 (term (,e0 ,e1 ,e1)))
  (module+ test
    (test-equal (redex-match? Lambda-calculus s s1) #true)
    (test-->> s-->comm #:equiv =α/racket
              (term (,p0))
              (term (1 5 (void)))))

  (define s-->comm
    (reduction-relation
     Lambda-calculus
     (--> (e_1 ... (in-hole E ((lambda (x ..._n) e) v ..._n)) e_2 ...)
          (e_1 ... (in-hole E (subst ([v x] ...) e)) e_2 ...)
          βv)
     (--> (e_1 ... (in-hole E (spawn e)) e_2 ...)
          (e_1 ... (in-hole E (void)) e e_2 ...)
          spawn)
     (--> (e_1 ... (in-hole E (get x)) e_2 ... (in-
hole E (put x v)) e_3 ...)
          (e_1 ... (in-hole E v) e_2 ... (in-hole E (void)) e_3 ...)
          message-left)
     (--> (e_1 ... (in-hole E (put x v)) e_2 ... (in-
hole E (get x)) e_3 ...)
          (e_1 ... (in-hole E v) e_2 ... (in-hole E (void)) e_3 ...)
          message-right)
     (--> (e_1 ... (in-hole E (+ n_1 n_2)) e_2 ...)
          (e_1 ... (in-hole E ,(+ (term n_1) (term n_2))) e_2 ...)
          +)
     (--> (e_1 ... (in-hole E (if0 0 e_then e_else)) e_2 ...)
          (e_1 ... (in-hole E e_then) e_2 ...)
          if0-true)
     (--> (e_1 ... (in-hole E (if0 v e_then e_else)) e_2 ...)
          (e_1 ... (in-hole E e_then) e_2 ...)
          (where #false (zero? (term v)))
          if0-false)))

  (module+ main
    (traces s-->comm s1))

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-results))
```

```
;;; ---------------------------------------------------------
;;; common.rkt starts here

#lang racket

;; basic definitions for the Redex Summer School 2015

(provide
 ;; Language
 Lambda

 ;; Any -> Boolean
 ;; is the given value in the expression language?
 lambda?

 ;; x (x ...) -> Boolean
 ;; (in x (x_1 ...)) determines whether x occurs in x_1 ...
 in

 ;; Any Any -> Boolean
 ;; (=α/racket e_1 e_2) determines whether e_1 is α-
equivalent to e_2
 ;; e_1, e_2 are in Lambda or extensions of Lambda that
 ;; do not introduce binding constructs beyond lambda
 =α/racket

 ;; ((Lambda x) ...) Lambda -> Lambda
 ;; (subs ((e_1 x_1) ...) e) substitures e_1 for x_1 ... in e
 ;; e_1, ... e are in Lambda or extensions of Lambda that
 ;; do not introduce binding constructs beyond lambda
 subst)

;; ---------------------------------------------------------------
-------------
(require redex)

(define-language Lambda
  (e ::=
     x
     (lambda (x_!_ ...) e)
     (e e ...))
  (x ::= variable-not-otherwise-mentioned))

(define lambda? (redex-match? Lambda e))
```

```
  (module+ test
    (define e1 (term y))
    (define e2 (term (lambda (y) y)))
    (define e3 (term (lambda (x y) y)))
    (define e4 (term (,e2 e3)))

    (test-equal (lambda? e1) #true)
    (test-equal (lambda? e2) #true)
    (test-equal (lambda? e3) #true)
    (test-equal (lambda? e4) #true)

    (define eb1 (term (lambda (x x) y)))
    (define eb2 (term (lambda (x y) 3)))

    (test-equal (lambda? eb1) #false)
    (test-equal (lambda? eb2) #false))

  ;; ---------------------------------------------------------------
-------------
  ;; (in x x_1 ...) is x a member of (x_1 ...)?

  (module+ test
    (test-equal (term (in x (y z x y z))) #true)
    (test-equal (term (in x ())) #false)
    (test-equal (term (in x (y z w))) #false))

  (define-metafunction Lambda
    in : x (x ...) -> boolean
    [(in x (x_1 ... x x_2 ...)) #true]
    [(in x (x_1 ...)) #false])

  ;; ---------------------------------------------------------------
-------------
  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

  (module+ test
    (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
    (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
    (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

  (define-metafunction Lambda
    =α : any any -> boolean
    [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

  ;; a Racket definition for use in Racket positions
  (define (=α/racket x y) (term (=α ,x ,y)))
```

```
;; (sd e) computes the static distance version of e
(define-extended-language SD Lambda
  (e ::= .... (K n))
  (n ::= natural))

(define SD? (redex-match? SD e))

(module+ test
  (define sd1 (term (K 1)))
  (define sd2 (term 1))

  (test-equal (SD? sd1) #true))

(define-metafunction SD
  sd : any -> any
  [(sd any_1) (sd/a any_1 ())])

(module+ test
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
  (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
              (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
  (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
              (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
  (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
              (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

(define-metafunction SD
  sd/a : any ((x ...) ...) -> any
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ;; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
  [(sd/a (lambda (x ...) any_1) (any_rest ...))
   (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
  [(sd/a (any_fun any_arg ...) (any_rib ...))
   ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
  [(sd/a any_1 any)
   ;; free variable, constant, etc
   any_1])


;; ---------------------------------------------------------------
```

```
-------------
  ;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)

  (module+ test
    (test-equal (term (subst ([1 x][2 y]) x)) 1)
    (test-equal (term (subst ([1 x][2 y]) y)) 2)
    (test-equal (term (subst ([1 x][2 y]) z)) (term z))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
                (term (lambda (z w) (1 2))))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
                (term (lambda (z w) (lambda (x) (x 2))))
                #:equiv =α/racket)
    (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
                (term ((lambda (x) (1 x)) 2))
                #:equiv =α/racket)
    (test-equal (term (subst (((lambda (x) y) x)) (lambda (y) x)))
                (term (lambda (y1) (lambda (x) y)))
                #:equiv =α/racket))

  (define-metafunction Lambda
    subst : ((any x) ...) any -> any
    [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
    [(subst [(any_1 x_1) ... ] x) x]
    [(subst [(any_1 x_1) ... ] (lambda (x ...) any_body))
     (lambda (x_new ...)
       (subst ((any_1 x_1) ...)
              (subst-raw ((x_new x) ...) any_body)))
     (where  (x_new ...)  ,(variables-not-in (term (any_body any_1 ...)) (term (x ...))))]
    [(subst [(any_1 x_1) ... ] (any ...)) ((subst [(any_1 x_1) ... ] any) ...)]
    [(subst [(any_1 x_1) ... ] any_*) any_*])

  (define-metafunction Lambda
    subst-raw : ((x x) ...) any -> any
    [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
    [(subst-raw ((x_n1 x_o1) ... ) x) x]
    [(subst-raw ((x_n1 x_o1) ... ) (lambda (x ...) any))
     (lambda (x ...) (subst-raw ((x_n1 x_o1) ... ) any))]
    [(subst-raw [(any_1 x_1) ... ] (any ...))
     ((subst-raw [(any_1 x_1) ... ] any) ...)]
    [(subst-raw [(any_1 x_1) ... ] any_*) any_*])

  ;; ---------------------------------------------------------------
-------------
  (module+ test
    (test-results))
```

## 3.15 Problem: Contracts

Design a reduction semantics (standard or regular) for a Lambda language with contracts. Here is the syntax:

```
(define-language Lambda
  (e ::=
     x (lambda (x) e) (e e)
     n (+ e e)
     (if0 e e e)
     (c · e x x)
     (blame x))
  (n ::= number)
  (c ::= num? even? odd? pos? (c -> c))
  (x ::= variable-not-otherwise-mentioned))
```

The contract primitives are interpreted as follows:

- `(num? x)` checks whether `x` is a number, not a function
- `(pos? x)` checks whether `x` is a positive number
- `(even? x)` checks whether `x` is an even number
- `(odd? x)` checks whether `x` is an even number

The contract form `(c · e x_s x_c)` checks contract `c` on `e`. If `e` breaks the contract, the semantics signals a `(blame x_s)` error; other contract violations signal a `(blame x_c)` error.

Consider these three examples where the same contracted function works well, is blamed, or blames its context depending on the argument:

```
(define a-module (term {(even? -> pos?) · (lambda (x) (+ x 1)) server client}))
(define p-good (term [,a-module 2]))
(define p-bad-server (term [,a-module -2]))
(define p-bad-client (term [,a-module 1]))
```

Work through the examples by hand to find out why the three programs work fine, blame the server, and blame the client for contract violations, respectively.

## 3.16 Solution: Contracts

```
#lang racket
```

```
  ;; a model of contracts in a call-by-value functional language

  (require redex "common.rkt")

  ;; ---------------------------------------------------------------------
-------------
  ;; syntax
  (define-language Lambda
    (e ::=
       x (lambda (x) e) (e e)
       n (+ e e)
       (if0 e e e)
       (c · e x x) ;; monitor a contract
       (blame x))
    (n ::= number)
    (c ::= num? even? odd? pos? (c -> c))
    (x ::= variable-not-otherwise-mentioned))

  ;; ---------------------------------------------------------------------
-------------
  ;; examples

  (define a-module (term {(even? -> pos?) · (lambda (x) (+ x 1)) server client}))
  (define p-good (term [,a-module 2]))
  (define p-bad-server (term [,a-module -2]))
  (define p-bad-client (term [,a-module 1]))

  (module+ test
    (test-equal (redex-match? Lambda c (term (even? -> pos?))) #t)
    (test-equal (redex-match? Lambda e p-good) #true)
    (test-equal (redex-match? Lambda e
                              (term
                               {(even? -> pos?) · (lambda (x) (+ x 1))
                                                  server
                                                  client})) #true)

    (test-equal (redex-match? Lambda e p-bad-server) #true)
    (test-equal (redex-match? Lambda e p-bad-client) #true))

  ;; ---------------------------------------------------------------------
-------------
  ;; the standard reductions

  (define-extended-language Lambda-calculus Lambda
    (v ::= n (lambda (x) e))
```

```
    (E ::= hole
       (v ... E e ...)
       (+ v ... E e ...)
       (c · E x x)))

  (module+ test
    (test-->> s-->c #:equiv =α/racket p-good 3)
    (test-->> s-->c #:equiv =α/racket p-bad-client (term (blame client)))
    (test-->> s-->c #:equiv =α/racket p-bad-server (term (blame server))))

  (define s-->c
    (reduction-relation
     Lambda-calculus
     (--> (in-hole E ((lambda (x) e) v)) (in-hole E (subst ([v x]) e)) βv)
     (--> (in-hole E (+ n_1 n_2)) (in-hole E ,(+ (term n_1) (term n_2))) +)
     (--> (in-hole E (if0 0 e_then e_else)) (in-hole E e_then) if0-
true)
     (--> (in-hole E (if0 v e_then e_else))
          (in-hole E e_then)
          (where #false (zero? (term v)))
          if0-false)
     (--> (in-hole E (pos? · n x_s x_c))
          (in-hole E ,(c positive? (term n) (term x_s) (term x_c)))
          pos)
     (--> (in-hole E (even? · n x_s x_c))
          (in-hole E ,(c even? (term n) (term x_s) (term x_c)))
          even)
     (--> (in-hole E (odd? · n x_s x_c))
          (in-hole E ,(c odd? (term n) (term x_s) (term x_c)))
          odd)
     (--> (in-hole E (num? · n x_s x_c))
          (in-hole E 0)
          num)
     (--> (in-hole E ((c_1 -> c_2) · (lambda (x) e) x_s x_c))
          (in-hole E
                   (lambda (x)
                     (c_2 · ((lambda (x) e) (c_1 · x x_c x_s)) x_s x_c))))
     (--> (in-hole E (blame x))
          (blame x)
          (where #false ,(equal? (term hole) (term E)))
          blame)))

  (define (c pred? n server client)
    (if (pred? n) n (term (blame ,server))))

  #;
```

```
  (module+ test
    (traces -->βv p-bad-client))

  ;; ------------------------------------------------------------
-------------
  (module+ test
    (test-results))



  ;;; ------------------------------------------------------------
  ;;; common.rkt starts here

  #lang racket

  ;; basic definitions for the Redex Summer School 2015

  (provide
   ;; Language
   Lambda

   ;; Any -> Boolean
   ;; is the given value in the expression language?
   lambda?

   ;; x (x ...) -> Boolean
   ;; (in x (x_1 ...)) determines whether x occurs in x_1 ...
   in

   ;; Any Any -> Boolean
   ;; (=α/racket e_1 e_2) determines whether e_1 is α-
equivalent to e_2
   ;; e_1, e_2 are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   =α/racket

   ;; ((Lambda x) ...) Lambda -> Lambda
   ;; (subs ((e_1 x_1) ...) e) substitures e_1 for x_1 ... in e
   ;; e_1, ... e are in Lambda or extensions of Lambda that
   ;; do not introduce binding constructs beyond lambda
   subst)

  ;; ------------------------------------------------------------
-------------
  (require redex)
```

```
  (define-language Lambda
    (e ::=
       x
       (lambda (x_!_ ...) e)
       (e e ...))
    (x ::= variable-not-otherwise-mentioned))

  (define lambda? (redex-match? Lambda e))

  (module+ test
    (define e1 (term y))
    (define e2 (term (lambda (y) y)))
    (define e3 (term (lambda (x y) y)))
    (define e4 (term (,e2 e3)))

    (test-equal (lambda? e1) #true)
    (test-equal (lambda? e2) #true)
    (test-equal (lambda? e3) #true)
    (test-equal (lambda? e4) #true)

    (define eb1 (term (lambda (x x) y)))
    (define eb2 (term (lambda (x y) 3)))

    (test-equal (lambda? eb1) #false)
    (test-equal (lambda? eb2) #false))

  ;; ---------------------------------------------------------------------
-------------
  ;; (in x x_1 ...) is x a member of (x_1 ...)?

  (module+ test
    (test-equal (term (in x (y z x y z))) #true)
    (test-equal (term (in x ())) #false)
    (test-equal (term (in x (y z w))) #false))

  (define-metafunction Lambda
    in : x (x ...) -> boolean
    [(in x (x_1 ... x x_2 ...)) #true]
    [(in x (x_1 ...)) #false])

  ;; ---------------------------------------------------------------------
-------------
  ;; (=α e_1 e_2) determines whether e_1 and e_2 are α equivalent

  (module+ test
```

```
  (test-equal (term (=α (lambda (x) x) (lambda (y) y))) #true)
  (test-equal (term (=α (lambda (x) (x 1)) (lambda (y) (y 1)))) #true)
  (test-equal (term (=α (lambda (x) x) (lambda (y) z))) #false))

(define-metafunction Lambda
  =α : any any -> boolean
  [(=α any_1 any_2) ,(equal? (term (sd any_1)) (term (sd any_2)))])

;; a Racket definition for use in Racket positions
(define (=α/racket x y) (term (=α ,x ,y)))

;; (sd e) computes the static distance version of e
(define-extended-language SD Lambda
  (e ::= .... (K n))
  (n ::= natural))

(define SD? (redex-match? SD e))

(module+ test
  (define sd1 (term (K 1)))
  (define sd2 (term 1))

  (test-equal (SD? sd1) #true))

(define-metafunction SD
  sd : any -> any
  [(sd any_1) (sd/a any_1 ())])

(module+ test
  (test-equal (term (sd/a x ())) (term x))
  (test-equal (term (sd/a x ((y) (z) (x)))) (term (K 2 0)))
  (test-equal (term (sd/a ((lambda (x) x) (lambda (y) y)) ()))
              (term ((lambda () (K 0 0)) (lambda () (K 0 0)))))
  (test-equal (term (sd/a (lambda (x) (x (lambda (y) y))) ()))
              (term (lambda () ((K 0 0) (lambda () (K 0 0))))))
  (test-equal (term (sd/a (lambda (z x) (x (lambda (y) z))) ()))
              (term (lambda () ((K 0 1) (lambda () (K 1 0)))))))

(define-metafunction SD
  sd/a : any ((x ...) ...) -> any
  [(sd/a x ((x_1 ...) ... (x_0 ... x x_2 ...) (x_3 ...) ...))
   ;; bound variable
   (K n_rib n_pos)
   (where n_rib ,(length (term ((x_1 ...) ...))))
   (where n_pos ,(length (term (x_0 ...))))
   (where #false (in x (x_1 ... ...)))]
```

```
    [(sd/a (lambda (x ...) any_1) (any_rest ...))
     (lambda () (sd/a any_1 ((x ...) any_rest ...)))]
    [(sd/a (any_fun any_arg ...) (any_rib ...))
     ((sd/a any_fun (any_rib ...)) (sd/a any_arg (any_rib ...)) ...)]
    [(sd/a any_1 any)
     ;; free variable, constant, etc
     any_1])


  ;; ---------------------------------------------------------------------
-------------
  ;; (subst ([e x] ...) e_*) substitutes e ... for x ... in e_* (hygienically)

  (module+ test
    (test-equal (term (subst ([1 x][2 y]) x)) 1)
    (test-equal (term (subst ([1 x][2 y]) y)) 2)
    (test-equal (term (subst ([1 x][2 y]) z)) (term z))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (x y))))
                (term (lambda (z w) (1 2))))
    (test-equal (term (subst ([1 x][2 y]) (lambda (z w) (lambda (x) (x y)))))
                (term (lambda (z w) (lambda (x) (x 2))))
                #:equiv =α/racket)
    (test-equal (term (subst ((2 x)) ((lambda (x) (1 x)) x)))
                (term ((lambda (x) (1 x)) 2))
                #:equiv =α/racket)
    (test-equal (term (subst (((lambda (x) y) x)) (lambda (y) x)))
                (term (lambda (y1) (lambda (x) y)))
                #:equiv =α/racket))

  (define-metafunction Lambda
    subst : ((any x) ...) any -> any
    [(subst [(any_1 x_1) ... (any_x x) (any_2 x_2) ...] x) any_x]
    [(subst [(any_1 x_1) ... ] x) x]
    [(subst [(any_1 x_1) ... ] (lambda (x ...) any_body))
     (lambda (x_new ...)
       (subst ((any_1 x_1) ...)
              (subst-raw ((x_new x) ...) any_body)))
     (where  (x_new ...)  ,(variables-not-in (term (any_body any_1 ...)) (term (x ...))))]
    [(subst [(any_1 x_1) ... ] (any ...)) ((subst [(any_1 x_1) ... ] any) ...)]
    [(subst [(any_1 x_1) ... ] any_*) any_*])

  (define-metafunction Lambda
    subst-raw : ((x x) ...) any -> any
    [(subst-raw ((x_n1 x_o1) ... (x_new x) (x_n2 x_o2) ...) x) x_new]
    [(subst-raw ((x_n1 x_o1) ... ) x) x]
    [(subst-raw ((x_n1 x_o1) ... ) (lambda (x ...) any))
```

```
     (lambda (x ...) (subst-raw ((x_n1 x_o1) ... ) any))]
    [(subst-raw [(any_1 x_1) ... ] (any ...))
     ((subst-raw [(any_1 x_1) ... ] any) ...)]
    [(subst-raw [(any_1 x_1) ... ] any_*) any_*])

  ;; ---------------------------------------------------------------------
-------------
  (module+ test
    (test-results))
```

## 3.17 Problem: Binary Addition

Redex can also model hardware scenarios.

Here is a language of *bit* expressions:

```
(define-language L
  (e ::=
     (or e e)
     (and e e)
     (not e)
     (append e ...)
     (add e e)
     v)
  (v ::= (b ...))
  (b ::= 0 1)
  (n ::= natural))
```

Your task is to design a standard reduction system that mimics addition on sequences of bits (binary digits).

**Hints** Raw values are sequences of booleans, not just one. For `or`, you can reduce sequences that have more than one element into `append`s of `or`s that each have a single element, and then actually handle `or`'s logic for arrays that have just one element. This also works for `and` and `not`. This idea doesn't work for `add`.

Here are some test cases to get you started:

```
(module+ test
 (test-->> red (term (or (1 1 0 0) (0 1 0 1))) (term (1 1 0 1)))
 (test-->> red (term (not (0 1))) (term (1 0)))
 (test-->> red (term (append (1 0) (0 1))) (term (1 0 0 1)))

 (test-->> red (term (or (1 1 0 0) (0 1 0 1))) (term (1 1 0 1)))
```

```
(test-->> red (term (and (1 1) (0 1))) (term (0 1)))
(test-->> red (term (and (0 0) (0 1))) (term (0 0))))
```

For testing add, we suggest comparing it to Racket's + operation using this helper function:

```
; v -> n (in L)
; convert a sequence of bits to a natural number

(module+ test
  (test-equal (to-nat (term ())) 0)
  (test-equal (to-nat (term (0))) 0)
  (test-equal (to-nat (term (1))) 1)
  (test-equal (to-nat (term (0 1))) 1)
  (test-equal (to-nat (term (1 0))) 2)
  (test-equal (to-nat (term (1 1))) 3)
  (test-equal (to-nat (term (1 1 1))) 7)
  (test-equal (to-nat (term (0 1 1 1))) 7)
  (test-equal (to-nat (term (0 1 1 0))) 6))

(define (to-nat bs)
  (for/sum ([b (in-list (reverse bs))]
            [i (in-naturals)])
    (* b (expt 2 i))))
```

## 3.18 Solution: Binary Addition

```
#lang racket

;; a model of hardware addition of bit sequences

(require redex)

(define-language L
  (e ::=
     (or e e)
     (and e e)
     (not e)
     (append e ...)
     (add e e)
     v)
  (v ::= (b ...))
  (n ::= natural)
  (b ::= 0 1))
```

```
(define red
  (compatible-closure
   (reduction-relation
    L

    (--> (or (b) (1)) (1) "or-1b")
    (--> (or (1) (b)) (1) "or-b1")
    (--> (or (0) (0)) (0) "or-00")

    (--> (or () ()) () "or-0")
    (--> (or (b_1 b_2 b_3 ...)
             (b_4 b_5 b_6 ...))
         (append (or (b_1) (b_4))
                 (or (b_2) (b_5))
                 (or (b_3) (b_6)) ...)
         "or-n")


    (--> (not (0)) (1) "not-1")
    (--> (not (1)) (0) "not-0")

    (--> (not (b_1 b_2 b_3 ...))
         (append (not (b_1))
                 (not (b_2))
                 (not (b_3)) ...)
         "not-n")
    (--> (not ()) () "not0")

    (--> (append (b ...)) (b ...) "append1")
    (--> (append (b_1 ...) (b_2 ...) (b_3 ...) ...)
         (append (b_1 ... b_2 ...) (b_3 ...) ...)
         "append2")

    (--> (and (b_1 ...) (b_2 ...))
         (not (or (not (b_1 ...))
                  (not (b_2 ...))))
         "and")

    (--> (add () (b ...)) (b ...))
    (--> (add (b ...) ()) (b ...))
    (--> (add (b ... 0) (b_2 ... b_1))
         (append (add (b ...) (b_2 ...)) (b_1)))
    (--> (add (b_2 ... b_1) (b ... 0))
         (append (add (b ...) (b_2 ...)) (b_1)))
    (--> (add (b_1 ... 1) (b_2 ... 1))
         (append (add (add (b_1 ...) (b_2 ...)) (1)) (0))))
```

```
   L e))

(module+ test
  (test-->> red (term (or (1 1 0 0) (0 1 0 1))) (term (1 1 0 1)))
  (test-->> red (term (not (0 1))) (term (1 0)))
  (test-->> red (term (append (1 0) (0 1))) (term (1 0 0 1)))

  (test-->> red (term (or (1 1 0 0) (0 1 0 1))) (term (1 1 0 1)))
  (test-->> red (term (and (1 1) (0 1))) (term (0 1)))
  (test-->> red (term (and (0 0) (0 1))) (term (0 0))))

;; rewrite-and-compare : (b ...) (b ...) -> boolean
(define (rewrite-and-compare b1s b2s)
  (define rewrite-answer
    (car
     (apply-reduction-relation*
      red
      (term (add ,b1s ,b2s)))))
  (if (redex-match? L (b ...) rewrite-answer)
      (equal? (+ (to-nat b1s) (to-nat b2s))
              (to-nat rewrite-answer))
      #f))

(define (to-nat bs)
  (for/sum ([b (in-list (reverse bs))]
            [i (in-naturals)])
    (* b (expt 2 i))))

(module+ test
  (test-equal (to-nat (term ())) 0)
  (test-equal (to-nat (term (0))) 0)
  (test-equal (to-nat (term (1))) 1)
  (test-equal (to-nat (term (0 1))) 1)
  (test-equal (to-nat (term (1 0))) 2)
  (test-equal (to-nat (term (1 1))) 3)
  (test-equal (to-nat (term (1 1 1))) 7)
  (test-equal (to-nat (term (0 1 1 1))) 7)
  (test-equal (to-nat (term (0 1 1 0))) 6))


(module+ test
  (test-equal (term (2nat ())) 0)
  (test-equal (term (2nat (0))) 0)
  (test-equal (term (2nat (1))) 1)
  (test-equal (term (2nat (0 1))) 1)
  (test-equal (term (2nat (1 0))) 2)
```

```
  (test-equal (term (2nat (1 1))) 3)
  (test-equal (term (2nat (1 1 1))) 7)
  (test-equal (term (2nat (0 1 1 1))) 7)
  (test-equal (term (2nat (0 1 1 0))) 6))

(define-metafunction L
  2nat : (b ...) -> natural
  [(2nat ()) 0]
  [(2nat (b_0 b_1 ...))
   ,(+ (term n_0) (term n_1))
   (where n_1 (2nat (b_1 ...)))
   (where n_0 ,(* (term b_0) (expt 2 (length (term (b_1 ...)))))])


;(traces red (term (and (1 1 0 0) (1 0 1 0))))

(module+ test
  (test-equal
   (for*/and ([b1 (in-list '(0 1))]
              [b2 (in-list '(0 1))]
              [b3 (in-list '(0 1))]
              [b4 (in-list '(0 1))]
              [b5 (in-list '(0 1))]
              [b6 (in-list '(0 1))])
     (rewrite-and-compare (list b1 b2 b3)
                          (list b4 b5 b6)))
   #t))

(module+ test (test-results))
```

# 4 The Redex Reference

```
(require redex)
```
package: redex-gui-lib

The redex library provides all of the names documented in this section.

Alternatively, use the redex/reduction-semantics and redex/pict libraries, which provide only non-GUI functionality (i.e., everything except redex/gui), making them suitable for programs that should not depend on racket/gui/base.

## 4.1 Patterns

```
(require redex/reduction-semantics)
```
package: redex-lib

This section covers Redex's *pattern* language, which is used in many of Redex's forms. Patterns are matched against terms, which are represented as S-expressions.

Pattern matching uses a cache—including caching the results of side-conditions—so after a pattern has matched a given term, Redex assumes that the pattern will always match the term.

In the following grammar, literal identifiers (such as any) are matched symbolically, as opposed to using the identifier's lexical binding:

```
pattern = any
        | _
        | number
        | natural
        | integer
        | real
        | string
        | boolean
        | variable
        | (variable-except id ...)
        | (variable-prefix id)
        | variable-not-otherwise-mentioned
        | hole
        | symbol
        | (name id pattern)
        | (in-hole pattern pattern)
        | (hide-hole pattern)
        | (side-condition pattern guard-expr)
        | (compatible-closure-context id)
        | (compatible-closure-context id #:wrt id)
        | (cross id)
```

```
                  | (pattern-sequence ...)
                  | other-literal

pattern-sequence = pattern
                  | ... ; literal ellipsis
                  | ..._id
```

- The `any` pattern matches any term. This pattern may also be suffixed with an underscore and another identifier, in which case a match binds the full name (as if it were an implicit `name` pattern) and match the portion before the underscore.

- The `_` pattern matches any term, but does not bind `_` as a name, nor can it be suffixed to bind a name.

- The `number` pattern matches any number.

  The `number` identifier can be suffixed with an underscore and additional characters, in which case the pattern binds the full name (as if it were an implicit `name` pattern) when matching the portion before the underscore. For example, the pattern

  ```
  number_1
  ```

  matches the same as `number`, but it also binds the identifier `number_1` to the matching portion of a term.

  When the same underscore suffix is used for multiple instances if `number` within a larger pattern, then the overall pattern matches only when all of the instances match the same number.

- The `natural` pattern matches any exact non-negative integer. Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `integer` pattern matches any exact integer. Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `real` pattern matches any real number. Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `string` pattern matches any string. Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `boolean` pattern matches `#true` and `#false` (which are the same as `#t` and `#f`, respectively). Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `variable` pattern matches any symbol. Like `number`, this pattern can be suffixed with an underscore and additional characters to create a binding.

- The `variable-except` pattern matches any symbol except those listed in its argument. This pattern is useful for ensuring that reserved words in the language are not accidentally captured by variables.

- The `variable-prefix` pattern matches any symbol that begins with the given prefix.
- The `variable-not-otherwise-mentioned` pattern matches any symbol except those that are used as literals elsewhere in the language.
- The `hole` pattern matches anything when inside the first argument to an `in-hole` pattern. Otherwise, it matches only a hole.
- The *`symbol`* pattern stands for a literal symbol that must match exactly, unless it is the name of a non-terminal in a relevant language or contains an underscore.

  If *`symbol`* is a non-terminal, it matches any of the right-hand sides of the non-terminal. If the non-terminal appears twice in a single pattern, then the match is constrained to expressions that are the same, unless the pattern is part of a `define-language` definition or a contract (e.g., in `define-metafunction`, `define-judgment-form`, or `define-relation`) in which case there is no constraint. Also, the non-terminal will be bound in the expression in any surrounding `side-condition` patterns unless there the pattern is in a `define-language` definition.

  If *`symbol`* is a non-terminal followed by an underscore, for example `e_1`, it is implicitly the same as a name pattern that matches only the non-terminal, `(name e_1 e)` for the example. Accordingly, repeated uses of the same name are constrained to match the same expression.

  If the symbol is a non-terminal followed by `_!_`, for example `e_!_1`, it is also treated as a pattern, but repeated uses of the same pattern are constrained to be different. For example, this pattern:

  ```
  (e_!_1 e_!_1 e_!_1)
  ```

  matches lists of three `e`s, but where all three of them are distinct.

  If the `_!_` is used under the ellipsis then the ellipsis is effectively ignored while checking to see if the `e`s are different. For example, the pattern `(e_!_1 ...)` matches any sequence of `e`s, as long as they are all distinct. Also, unlike `e_1` patterns, the nesting depth of `_!_` patterns do not have to be the same. For example, this pattern:

  ```
  (e_!_1 ... e_!_1)
  ```

  matches all sequences of `e`s that have at least one element, as long as they are all distinct.

  Unlike a `_` pattern, the `_!_` patterns do not bind names.

  If `_` names and `_!_` are mixed, they are treated as separate. That is, this pattern `(e_1 e_!_1)` matches just the same things as `(e e)`, but the second doesn't bind any variables.

  If the symbol otherwise has an underscore, it is an error.
- The pattern `(name` *`id pattern`*`)` matches *`pattern`* and binds using it to the name *`id`*.

- The (in-hole *pattern pattern*) pattern matches the first *pattern*, looking for a way to decompose the term such that the second *pattern* matches at some sub-expression where the hole appears while matching the first *pattern*.

  The first *pattern* must be a pattern that matches with exactly one hole.

- The (hide-hole *pattern*) pattern matches what the embedded *pattern* matches but if the pattern matcher is looking for a decomposition, it ignores any holes found in that *pattern*.

- The (side-condition *pattern guard-expr*) pattern matches what the embedded *pattern* matches, and then *guard-expr* is evaluated. If *guard-expr* produces #f, the pattern fails to match, otherwise the pattern matches. Any occurrences of name in the pattern (including those implicitly present via _ patterns) are bound using term-let in *guard-expr*.

- The (compatible-closure-context *nt*) pattern matches context that correspond to where the compatible closure of a relation would match. More precisely, it is a context whose shape follows the definition of *nt*, but allowing for a hole at each place where the definition of *nt* refers to itself.

  For example, with this language definition:

  ```
  (define-language L
    (e ::= (λ (x) e) (e e) x)
    (C ::= (λ (x) C) (C e) (e C) hole)
    (x ::= variable-not-otherwise-mentioned))
  ```

  the pattern (compatible-closure-context e) is equivalent to the pattern C.

  The (compatible-closure-context *nt1* #:wrt *nt2*) pattern similarly is a context, but it decomposes terms matching the non-terminal *nt1*, placing a hole at each place where an *nt2* non-terminal appears.

  For example, with this language definition:

  ```
  (define-language L
    (e ::= v (e e) x)
    (v ::= (λ (x) e))
    (C ::= V (C e) (e C) hole)
    (V ::= (λ (x) C))
    (x ::= variable-not-otherwise-mentioned))
  ```

  the pattern (compatible-closure-context v #:wrt e) is equivalent to the pattern V and the pattern the pattern (compatible-closure-context e #:wrt e) is equivalent to the pattern C. More generally, leaving off the #:wrt argument is the same as using the same non-terminal twice.

- The (cross *nt*) pattern is an unfortunately-named version of compatible-closure-context that exists for backward compatibility and does not support #:wrt.

- The (*pattern-sequence* ...) pattern matches a term list, where each pattern-sequence element matches an element of the list. In addition, if a list pattern contains an ellipsis, the ellipsis is not treated as a literal, instead it matches any number of duplicates of the pattern that came before the ellipses (including 0). Furthermore, each (name symbol *pattern*) in the duplicated pattern binds a list of matches to symbol, instead of a single match. (A nested duplicated pattern creates a list of list matches, etc.) Ellipses may be placed anywhere inside the row of patterns, except in the first position or immediately after another ellipses.

  Multiple ellipses are allowed. For example, this pattern:

  ```
  ((name x a) ... (name y a) ...)
  ```

  matches this term:

  ```
  (term (a a))
  ```

  three different ways. One where the first a in the pattern matches nothing, and the second matches both of the occurrences of a, one where each named pattern matches a single a and one where the first matches both and the second matches nothing.

  If the ellipses is named (i.e., has an underscore and a name following it, like a variable may), the pattern matcher records the length of the list and ensures that any other occurrences of the same named ellipses must have the same length.

  As an example, this pattern:

  ```
  ((name x a) ..._1 (name y a) ..._1)
  ```

  only matches this term:

  ```
  (term (a a))
  ```

  one way, with each named pattern matching a single a. Unlike the above, the two patterns with mismatched lengths is ruled out, due to the underscores following the ellipses.

  Also, like underscore patterns above, if an underscore pattern begins with ..._!_, then the lengths must be different.

  Thus, with the pattern:

  ```
  ((name x a) ..._!_1 (name y a) ..._!_1)
  ```

  and the expression

  ```
  (term (a a))
  ```

  two matches occur, one where x is bound to '() and y is bound to '(a a) and one where x is bound to '(a a) and y is bound to '().

- The *other-literal* pattern stands for a literal value—such as a number, boolean, or string—that must match exactly.

Changed in version 1.8 of package `redex-lib`: Non-terminals are syntactically classified as either always producing exactly one hole or may produce some other number of holes, and the first argument to `in-hole` is allowed to accept only patterns that produce exactly one hole.
Changed in version 1.15: Added `compatible-closure-context`

```
(redex-match lang pattern term-expr)
(redex-match lang pattern)
```

If `redex-match` is given a *term-expr*, it matches the pattern (in the language) against the result of *term-expr*. The result is `#f` or a list of match structures describing the matches (see `match?` and `match-bindings`).

If `redex-match` has only a *lang* and *pattern*, the result is a procedure for efficiently testing whether terms match the pattern with respect to the language *lang*. The procedure accepts a single term and returns `#f` or a list of match structures describing the matches.

Examples:

```
> (define-language nums
    (AE number
        (+ AE AE)))
> (redex-match nums
               (+ AE_1 AE_2)
               (term (+ (+ 1 2) 3)))
(list (match (list (bind 'AE_1 '(+ 1 2)) (bind 'AE_2 3))))
> (redex-match nums
               (+ AE_1 (+ AE_2 AE_3))
               (term (+ (+ 1 2) 3)))
#f
> (redex-match nums
               (+ AE_1 AE_1)
               (term (+ (+ 1 2) 3)))
#f
```

```
(redex-match? lang pattern any)
(redex-match? lang pattern)
```

Like `redex-match`, but returns only a boolean indicating whether the match was successful.

Examples:

```
> (define-language nums
    (AE number
        (+ AE AE)))
```

```
> (redex-match? nums
                (+ AE_1 AE_2)
                (term (+ (+ 1 2) 3)))
#t
> (redex-match? nums
                (+ AE_1 AE_1)
                (term (+ (+ 1 2) 3)))
#f
```

```
(match? val) → boolean?
  val : any/c
```

Determines whether a value is a `match` structure.

```
(match-bindings m) → (listof bind?)
  m : match?
```

Returns a list of `bind` structs that binds the pattern variables in this match.

```
(struct bind (name exp)
    #:extra-constructor-name make-bind)
  name : symbol?
  exp : any/c
```

Instances of this struct are returned by `redex-match`. Each `bind` associates a name with an s-expression from the language, or a list of such s-expressions if the corresponding `name` clause is followed by an ellipsis. Nested ellipses produce nested lists.

```
(caching-enabled?) → boolean?
(caching-enabled? on?) → void?
  on? : boolean?
```

When this parameter is `#t` (the default), Redex caches the results of pattern matching, metafunction, and judgment-form evaluation. There is a separate cache for each pattern, metafunction, and judgment-form; when one fills (see `set-cache-size!`), Redex evicts all of the entries in that cache.

Caching should be disabled when matching a pattern that depends on values other than the in-scope pattern variables or evaluating a metafunction or judgment-form that reads or writes mutable external state.

Changed in version 1.6 of package `redex-lib`: Extended caching to cover judgment forms.

```
(set-cache-size! size) → void?
  size : positive-integer?
```

Changes the size of the per-pattern, per-metafunction and per-judgment-form caches.

The default size is 63.

```
(check-redundancy) → boolean?
(check-redundancy check?) → void?
  check? : boolean?
```

Ambiguous patterns can slow down Redex's pattern matching implementation significantly. To help debug such performance issues, set the `check-redundancy` parameter to `#t`. A true value causes Redex to, at runtime, report any redundant matches that it encounters.

Changed in version 1.9 of package `redex-lib`: Corrected spelling error, from `check-redudancy` to `check-redundancy`

## 4.2 Terms

Object language expressions in Redex are written using `term`. It is similar to Racket's `quote` (in many cases it is identical) in that it constructs lists as the visible representation of terms.

The grammar of *term*s is (note that an ellipsis stands for repetition unless otherwise indicated):

```
           term = identifier
                | (term-sequence ...)
                | ,expr
                | (in-hole term term)
                | hole
                | (mf-apply identifier term ...)
                | datum

  term-sequence = term
                | ,@expr
                | ... ; literal ellipsis
```

- A term written *identifier* is equivalent to the corresponding symbol, unless the identifier is bound by `term-let`, `term-define`, `define-term`, or a pattern variable or the identifier is `hole` (as below).
- A term written `(`*term-sequence* `...)` constructs a list of the terms constructed by the sequence elements.
- A term written `,`*expr* evaluates *expr* and substitutes its value into the term at that point.

- A term written ,@*expr* evaluates the *expr*, which must produce a list. It then splices the contents of the list into the expression at that point in the sequence.
- A term written (in-hole *term* *term*) is the dual to the pattern in-hole – it accepts a context and an expression and uses plug to combine them.
- A term written hole produces a hole.
- A term written (mf-apply f arg ...) asserts that f is a metafunction and produces the term (f arg ...).
- A term written as any other datum not listed above produces that datum. For example, (term (1 x #t)) is the same as '(1 x #t).

  Term substitution and metafunction application do not occur within compound datums. For example,

  ```
  (term-let ([a 1]) (term #hash((x . a))))
  ```

  is the same as '#hash((x . a)), not '#hash((x . 1)).

```
(term term)
(term term #:lang lang-id)
```

Used for construction of a term.

The term form behaves similarly to quasiquote, except for a few special forms that are recognized (listed below) and that names bound by term-let and term-define are implicitly substituted with the values that those names were bound to, expanding ellipses as in-place sublists (in the same manner as syntax-case patterns).

The optional #:lang keyword must supply an identifier bound by define-language, and adds a check that any symbol containing an underscore in *term* could have been bound by a pattern in the language referenced by *lang-id*. In practice, this means that the underscore must be preceded by a non-terminal in that language or a built-in *pattern* such as number. This form of term is used internally by default, so this check is applied to terms that are constructed by Redex forms such as reduction-relation and define-metafunction.

For example,

```
(term-let ([body '(+ x 1)]
           [(expr ...) '(+ - (values * /))]
           [((id ...) ...) '((a) (b) (c d))])
  (term (let-values ([(id ...) expr] ...) body)))
```

evaluates to

```
'(let-values ([(a) +]
              [(b) -]
              [(c d) (values * /)])
   (+ x 1))
```

It is an error for a term variable to appear in an expression with an ellipsis-depth different from the depth with which it was bound by `term-let`. It is also an error for two `term-let`-bound identifiers bound to lists of different lengths to appear together inside an ellipsis.

Symbols in a term whose names end in guillemets (French quotes) around a number (for example `asdf«5000»`) will be modified to contain a smiley face character (for example `asdf«5000☺»`). This is to prevent collisions with names generated by the freshening process that binding forms use.

`hole`

Recognized specially within terms. A `hole` form is an error elsewhere.

`in-hole`

Recognized specially within terms. An `in-hole` form is an error elsewhere.

`mf-apply`

Recognized specially within terms. A `mf-apply` form is an error elsewhere.

```
(term-let ([tl-pat expr] ...) body)

    tl-pat = identifier
           | (tl-pat-ele ...)

tl-pat-ele = tl-pat
           | tl-pat ... ; a literal ellipsis
```

Matches each given id pattern to the value yielded by evaluating the corresponding expression and binds each variable in the id pattern to the appropriate value (described below). These bindings are then accessible to the *term* syntactic form.

Note that each ellipsis should be the literal symbol consisting of three dots (and the ... elsewhere indicates repetition as usual). If *tl-pat* is an identifier, it matches any value and binds it to the identifier, for use inside `term`. If it is a list, it matches only if the value being matched is a list value and only if every subpattern recursively matches the corresponding list element. There may be a single ellipsis in any list pattern; if one is present, the pattern before the ellipses may match multiple adjacent elements in the list value (possibly none).

This form is a lower-level form in Redex, and not really designed to be used directly. For `let`-like forms that use Redex's full pattern matching facilities, see `redex-let`, `redex-let*`, `term-match`, `term-match/single`.

Examples:

```
> (term-let ([x 1]
             [y #t]
             [z '(p q r)])
    (term (x y z)))
'(1 #t (p q r))
> (term-let ([((init ... last) ...) '((1 2 3 x) (4 5 y))])
    (term (last ...)))
'(x y)
> (term-let ([((init ... last) ...) '((1 2 3 x) (4 5 y))])
    (term ((~@ init init) ... ...)))
'(1 1 2 2 3 3 4 4 5 5)
```

```
(term-define tl-pat expr)
```

The `define` analog of `term-let`.

Examples:

```
> (term-define z '(p q r))
> (term z)
'(p q r)
> (term-define ((init ... last) ...)
    '((1 2 3 x) (4 5 y)))
> (term (last ...))
'(x y)
> (term ((~@ init init) ... ...))
'(1 1 2 2 3 3 4 4 5 5)
```

Added in version 1.21 of package `redex-lib`.

```
(redex-let language ([pattern expression] ...) body ...+)
```

Like `term-let` but the left-hand sides are Redex patterns, interpreted according to the specified language. It is a syntax error for two left-hand sides to bind the same pattern variable.

This form raises an exception recognized by `exn:fail:redex?` if any right-hand side does not match its left-hand side in exactly one way.

In some contexts, it may be more efficient to use `term-match/single` (lifted out of the context).

```
(redex-let* language ([pattern expression] ...) body ...+)
```

The `let*` analog of `redex-let`.

```
(redex-define language pattern expression)
```

The `define` analog of `redex-let`. The form `redex-define` evaluates *expression*, matches the result against *pattern* and binds the corresponding identifiers.

Examples:

```
> (define-language Larith
    (AE number
        (+ AE AE)))
> (redex-define Larith (name AE_all (+ AE_common AE_common)) (term (+ 4 4)))
> (term (AE_all AE_common))
'((+ 4 4) 4)
> (redex-define Larith AE_stx-err (term (+ (+ 0 #f) 1)))
redex-define: term (+ (+ 0 #f) 1) does not match pattern
AE_stx-err
> (redex-define Larith (+ AE_same AE_same) (term (+ 6 3)))
redex-define: term (+ 6 3) does not match pattern (+ AE_same
AE_same)
```

Examples:

```
> (define-language Lempty)
> (redex-define Lempty (number ...) (term (2 1 7)))
> (term ((~@ number number) ...))
'(2 2 1 1 7 7)
```

Added in version 1.21 of package `redex-lib`.

```
(define-term identifier term)
```

Defines *identifier* for use in term templates. To pattern match and bind identifiers against *term*, see `redex-define`.

```
(term-match language [pattern expression] ...)
```

Produces a procedure that accepts term (or quoted) expressions and checks them against each pattern. The function returns a list of the values of the expression where the pattern matches. If one of the patterns matches multiple times, the expression is evaluated multiple times, once with the bindings in the pattern for each match.

When evaluating a `term-match` expression, the patterns are compiled in an effort to speed up matching. Using the procedural result multiple times to avoid compiling the patterns multiple times.

```
(term-match/single language [pattern expression] ...)
```

Produces a procedure that accepts term (or quoted) expressions and checks them against each pattern. The function returns the expression behind the first successful match. If that pattern produces multiple matches, an error is signaled. If no patterns match, an error is signaled.

The `term-match/single` form raises an exception recognized by `exn:fail:redex?` if no clauses match or if one of the clauses matches multiple ways.

When evaluating a `term-match/single` expression, the patterns are compiled in an effort to speed up matching. Using the procedural result multiple times to avoid compiling the patterns multiple times.

```
(plug context expression) → any
  context : any/c
  expression : any/c
```

The first argument to this function is an term to plug into. The second argument is the term to replace in the first argument. It returns the replaced term. This is also used when a `term` sub-expression contains `in-hole`.

```
(variable-not-in t prefix) → symbol?
  t : any/c
  prefix : symbol?
```

A helper function that accepts a term and a variable. It returns a symbol that not in the term, where the variable has *prefix* as a prefix.

```
(variables-not-in t vars) → (listof symbol?)
  t : any/c
  vars : (listof symbol?)
```

Like `variable-not-in`, create variables that do no occur in *t*—but returning a list of such variables, one for each variable in its second argument.

The `variables-not-in` function does not expect the symbols in *vars* to be distinct, but it does produce a list of distinct symbols.

```
(exn:fail:redex? v) → boolean?
  v : any/c
```

Returns `#t` if its argument is a Redex exception record, and `#f` otherwise.

## 4.3 Languages

```
(define-language lang-name
  non-terminal-def ...
  maybe-binding-spec)

  non-terminal-def = (non-terminal-name ...+ ::= pattern ...+)
                   | (non-terminal-name pattern ...+)
                   | ((non-terminal-name ...+) pattern ...+)

maybe-binding-spec =
                   | #:binding-forms binding-pattern ...

   binding-pattern = pattern
                   | binding-pattern #:exports beta
                   | binding-pattern #:refers-to beta
                   | binding-pattern #:...bind (id beta beta)

              beta = nothing
                   | symbol
                   | (shadow beta-sequence ...)

     beta-sequence = beta
                   | ... ; literal ellipsis
```

Defines the grammar of a language. The `define-language` form supports the definition of recursive patterns, much like a BNF, but for regular-tree grammars. It goes beyond their expressive power, however, because repeated `name`, `in-hole`, and side-condition patterns can restrict matches in complex ways.

A *non-terminal-def* comprises one or more non-terminal names (considered aliases) followed by one or more productions.

For example, the following defines *lc-lang* as the grammar of the $\lambda$-calculus:

```
(define-language lc-lang
  (e ::= (e e ...)
         x
         (λ (x ...) e))
  (v ::= (λ (x ...) e))
  (E ::= (v ... E e ...)
         hole)
  (x y ::= variable-not-otherwise-mentioned))
```

It has non-terminals: `e` for the expression language, `x` and `y` for variables, `v` for values, and `E` for the evaluation contexts.

Non-terminals used in `define-language` are not bound in `side-condition` patterns. Duplicate non-terminals that appear outside of the binding-forms section are not constrained to be the same unless they have underscores in them.

### 4.3.1 Binding Forms

Typical languages provide a mechanism for the programmer to introduce new names and give them meaning. The language forms used for this (such as Racket's `let` and $\lambda$) are called *binding forms*.

Binding forms require special treatment from the language implementer. In Redex, this treatment consists of declaring the binding forms at the time of language definition. Explicitly declaring binding forms makes safely manipulating terms containing binding simpler and easier, eliminating the need to write operations that (explicitly) respect the binding structure of the language.

When *`maybe-binding-spec`* is provided, it declares binding specifications for certain forms in the language. The *`binding-pattern`* specification is an extension of Redex's pattern language, allowing the keywords `#:refers-to`, `#:exports`, and `#:...binds` to appear nested inside a binding pattern.

The language, *`lc-lang`*, above does not declare any binding specifications, despite the clear intention of $\lambda$ as a binding form. To understand the consequences of not specifying any binding forms, consider the behavior of substitution on terms of *`lc-lang`*.

```
> (term (substitute (x (λ (x) (λ (y) x)))
                    x
                    (y y)) #:lang lc-lang)
'((y y) (λ ((y y)) (λ (y) (y y))))
```

Passing the `#:lang` argument to `term` allows the `substitute` metafunction to determine the language of its arguments.

This call is intended to replace all free occurrences of `x` with `(y y)` in the first argument to `substitute`. But, because *`lc-lang`* is missing a binding forms declaration, `substitute` replaces all instances of `x` with `(y y)` in the term `(x (λ (x) (λ (y) x)))`. Note that even the `x` that appears in what is normally a binding position has been replaced, resulting in an ill-formed lambda expression.

In order to have `substitute` behave correctly when substituting over terms that contain bound variables, the language *`lc-lang`* must declare its binding specification. Consider the following simplification of the *`lc-lang`* definition, this time with a binding form declaration for $\lambda$.

```
(define-language lc-bind
  (e ::= (e e)
         x
         (λ (x) e))
```

```
  (v ::= (λ (x) e))
  (x y ::= variable-not-otherwise-mentioned)
  #:binding-forms
  (λ (x) e #:refers-to x))
```

Just like Racket's $\lambda$, in *lc-bind* all instances of the argument variable in the body of the lambda refer to the argument. In a binding declaration, this is specified using the `#:refers-to` keyword. Now the previous example has the right behavior.

```
> (term (substitute (x (λ (x) (λ (y) x)))
                    x
                    (y y)) #:lang lc-bind)
'((y y) (λ (x«0») (λ (y«1») x«0»)))
```

Note that sometimes substitute changes the names of the bound identifiers, in this case replacing the `x` and `y` with identifiers that have `«` and `»` in their names.

The `#:refers-to` declaration says that, in a $\lambda$ term, the `e` subterm has the name from the `x` subterm in scope.

### 4.3.2 Multiple Variables in a Single Scope

To generalize to the version of $\lambda$ in *lc-lang*, we need to cope with multiple variables at once. And in order to do that, we must handle the situation where some of the names are the same. Redex's binding support offers only one option for this, namely taking the variables in order, using the keyword *shadow*. It also allows us to specify the binding structure for `let`:

```
(define-language lc-bind+let
  (e ::= x
         number
         (λ (x ...) e)
         (e e)
         (let ([x e] ...) e))
  (x ::= variable-not-otherwise-mentioned)
  #:binding-forms
  (λ (x ...) e #:refers-to (shadow x ...))
  (let ([x e_x] ...) e_body #:refers-to (shadow x ...)))
```

This `#:binding-forms` declaration says that the subterm `e` of the $\lambda$ expression refers to all of the binders in $\lambda$. Similarly, the `e_body` refers to all of the binders in the `let` expression.

```
> (term (substitute (let ([x 5] [y x]) (y x))
```

```
                    x
                    z) #:lang lc-bind+let)
'(let ((x«2» 5) (y«3» z)) (y«3» x«2»))
```

The intuition behind the name of the `shadow` form can be seen in the following example:

```
> (term (substitute (let ([x 1] [y x] [x 3]) x)
                    x
                    z) #:lang lc-bind+let)
'(let ((x«4» 1) (y«5» z) (x«6» 3)) x«4»)
```

Because the *lc-bind+let* language does not require that all binders in its `let` form be distinct from one another, the binding forms specification must declare what happens when there is a conflict. The `shadow` form specifies that duplicate binders will be shadowed by earlier binders in its list of arguments. (Of course, if we were interested in modelling Racket's `let` form, we'd want that term to be malformed syntax.)

It is possible to have multiple uses of `#:refers-to` in a single binding specification. For example, consider a language with a `letrec` form.

```
(define-language lc-bind+letrec
  (e ::= x
         number
         (λ (x ...) e)
         (e e)
         (let ([x e] ...) e)
         (letrec ([x e] ...) e))
  (x ::= variable-not-otherwise-mentioned)
  #:binding-forms
  (λ (x ...) e #:refers-to (shadow x ...))
  (let ([x e_x] ...) e_body #:refers-to (shadow x ...))
  (letrec ([x e_x] ...) #:refers-to (shadow x ...) e_body #:refers-
to (shadow x ...)))
```

In this binding specification the subterms corresponding to both `([x e_x] ...)` and `e_body` refer to the bound variables `(shadow x ...)`.

```
> (term (substitute (letrec ([x x]) x) x y) #:lang lc-bind+letrec)
'(letrec ((x«7» x«7»)) x«7»)
```

```
> (term
   (substitute
    (letrec ([x (λ (a) (y a))]
             [y (λ (b) (z b))]
```

```
            [z a])
      (x 7))
    a
    (λ (x) 5))
   #:lang lc-bind+letrec)
'(letrec ((x«8» (λ (a«11») (y«9» a«11»)))
          (y«9» (λ (b«12») (z«10» b«12»)))
          (z«10» (λ (x) 5)))
   (x«8» 7))
```

### 4.3.3 Ellipses in Binding Forms

Some care must be taken when writing binding specifications that match patterns with ellipses. If a pattern symbol is matched underneath ellipses, it may only be mentioned under the same number of ellipses. Consider, for example, a language with Racket's `let-values` binding form.

```
(define-language lc-bind+values
  (e ::= x
         number
         (λ (x ...) e)
         (e e)
         (values e ...)
         (let-values ([(x ...) e] ...) e))
  (x ::= variable-not-otherwise-mentioned)
  #:binding-forms
  (λ (x ...) e #:refers-to (shadow x ...))
  (let-values ([(x ...) e_x0] ...)
    e_body #:refers-to (shadow (shadow x ...) ...)))
```

In the binding specification for the `let-values` form, the bound variable, `x`, occurs only under a single ellipsis, thus when it is mentioned in a `#:refers-to` clause it is restricted to be mentioned only underneath a single ellipsis. Therefore the body of the `let-values` form must refer to `(shadow (shadow x ...) ...)` rather than `(shadow x ... ...)`.

### 4.3.4 Compound Forms with Binders

So far, the nonterminals mentioned in `#:refers-to` have always stood directly for variables that appear in the terms. But sometimes the variables are down inside some piece of the term, or only some of the variables are relevant. The `#:exports` clause can be used to handle such situations.

When a binding form with an `#:exports` clause is mentioned, the names brought into scope

are determined by recursively examining everything mentioned by that `#:exports` clause. Consider the following version of the *lc-bind* language with lists that allows for pattern matching in binding positions.

```
(define-language lc-bind+patterns
    (e ::= x
           number
           (λ (p) e)
           (e e)
           (list e ...))
    (x ::= variable-not-otherwise-mentioned)
    (p ::= (listp p ...) x)
    #:binding-forms
    (λ (p) e #:refers-to p)
    (listp p ...) #:exports (shadow p ...))
```

In this language functions accept patterns as arguments, therefore all variables mentioned in a pattern in binding position should be bound in the body of the function. A call to the `substitute` metafunction shows this behavior.

```
> (term
   (substitute (x (λ ((listp w (listp x y) z)) (list z y x w)))
               x
               u)
   #:lang lc-bind+patterns)
'(u
  (λ ((listp w«13» (listp x«14» y«15») z«16»)) (list z«16» y«15»
x«14» w«13»)))
```

The use of the `#:exports` clause in the binding specification for *lc-bind+patterns* allows the use of nested binding patterns seen in the example. More precisely, each `p` may itself be a pattern that mentions any number of bound variables.

### 4.3.5 Binding Repetitions

In some situations, the `#:exports` and `#:refers-to` keywords are not sufficiently expressive to be able to describe the binding structure of different parts of a repeated sequence relate to each other. For example, consider the `let*` form. Its shape is the same as `let`, namely `(let* ([x e] ...) e)`, but the binding structure is different.

In a `let*` form, each variable is accessible to each of the `e`s that follow it, with all of the variables available in the body (the final `e`). With `#:exports`, we can build an expression form that has a structure like that, but we must write syntax that nests differently than `let*`.

```
(define-language lc-bind+awkward-let*
  (e ::= (let*-awk c e) natural x (+ e ...))
  (x ::= variable-not-otherwise-mentioned)
  (c ::= (clause x e c) ())
  #:binding-forms
  (let*-awk c e #:refers-to c)
  (clause x e c #:refers-to x) #:exports (shadow x c))
```

The `let*-awk` form binds like Racket's `let*`, with each clause's variable being active for the subsequent ones, but the syntax is different with extra nesting inside the clauses:

```
> (term (substitute (let*-awk (clause x y (clause y x ()))
                              (+ x y z))
                    x
                    1)
        #:lang lc-bind+awkward-let*)
'(let*-awk (clause x«17» y (clause y«18» x«17» ())) (+ x«17» y«18»
z))
> (term (substitute (let*-awk (clause x y (clause y x ()))
                              (+ x y z))
                    y
                    2)
        #:lang lc-bind+awkward-let*)
'(let*-awk (clause x«19» 2 (clause y«20» x«19» ())) (+ x«19» y«20»
z))
```

In order to get the same syntax as Racket's `let*`, we need to use the `#:...bind` binding pattern annotation. A `#:...bind` can appear wherever a `...` might appear, and it has the same function, namely indicating a repetition of the preceding pattern. In addition, however it comes with three extra pieces that follow the `#:...bind` form that describe how the binding structure inside the repetition is handled. The first part is a name that can be used by a `#:refers-to` outside of the repetition to indicate all of the exported variables of the sequence. The middle piece indicates the variables from a specific repetition of the ellipsis are exported to all subsequent repetitions of the ellipsis. The last piece is a *`beta`* that moves backwards through the sequence, indicating what is exported from the last repetition of the sequence to the one before, from the one before to the one before that, and then finally from the first one to the export of the entire sequence (as named by the identifier in the first position).

So, in this example, we use `#:...bind` to express the scope of `let*`.

```
(define-language lc-bind+let*
  (e ::= (let* ([x e] ...) e) natural x (+ e ...))
  (x ::= variable-not-otherwise-mentioned)
  #:binding-forms
```

```
(let* ([x e] #:...bind (clauses x (shadow clauses x)))
  e_body #:refers-to clauses))
```

It says that the name of the exported variables from the entire sequence is `clauses`, which means that all of the variable exported from the sequence in the second position of the `let*` bind variables in the body (thanks to the last `#:refers-to` in the example). The `x` in the second position following the `#:...bind` says that `x` is in scope for each of the subsequent `[x e]` elements of the sequence. The final `(shadow clauses x)` says that the variables in a subsequent `clauses` are exported by the current one, as well as `x`, which then is exported by the entire sequence.

```
> (term (substitute (let* ([x y] [y x])
                      (+ x y z))
                    x
                    1)
        #:lang lc-bind+let*)
'(let* ((x«22» y) (y«21» x«22»)) (+ x«22» y«21» z))
> (term (substitute (let* ([x y] [y x])
                      (+ x y z))
                    y
                    2)
        #:lang lc-bind+let*)
'(let* ((x«24» 2) (y«23» x«24»)) (+ x«24» y«23» z))
```

`::=`

A non-terminal's names and productions may be separated by the keyword `::=`. Use of the `::=` keyword outside a language definition is a syntax error.

`shadow`

Recognized specially within a `define-language`. A `shadow` is an error elsewhere.

`nothing`

Recognized specially within a `define-language`. A `nothing` is an error elsewhere.

```
(define-extended-language extended-lang base-lang
  non-terminal-def ...
  maybe-binding-spec)

  non-terminal-def = (non-terminal-name ...+ ::= pattern ...+)
                   | (non-terminal-name pattern ...+)
                   | ((non-terminal-name ...+) pattern ...+)

maybe-binding-spec =
                   | #:binding-forms binding-declaration ...
```

Extends a language with some new, replaced, or extended non-terminals. For example, this language:

```
(define-extended-language lc-num-lang
  lc-lang
  (e ::= ....      ; extend the previous `e' non-terminal
     number
     +)
  (v ::= ....      ; extend the previous `v' non-terminal
     number
     +))
```

extends *lc-lang* with two new alternatives (`+` and `number`) for the `v` non-terminal, carries forward the `e`, `E`, `x`, and `y` non-terminals. Note that the meaning of `variable-not-otherwise-mentioned` adapts to the language where it is used, so in this case it is equivalent to `(variable-except λ +)` because λ and `+` are used as literals in this language.

The four-period ellipses indicates that the new language's non-terminal has all of the alternatives from the original language's non-terminal, as well as any new ones. If a non-terminal occurs in both the base language and the extension, the extension's non-terminal replaces the originals. If a non-terminal only occurs in the base language, then it is carried forward into the extension. And, of course, `define-extended-language` lets you add new non-terminals to the language.

If a language has a group of multiple non-terminals defined together, extending any one of those non-terminals extends all of them.

```
(define-union-language L base/prefix-lang ...)

base/prefix-lang = lang-id
                 | (prefix lang-id)
```

Constructs a language that is the union of all of the languages listed in the *base/prefix-lang*.

If the two languages have non-terminals in common, then `define-union-language` will combine all of the productions of the common non-terminals. For example, this definition of *L*:

```
(define-language L1
  (e ::=
     (+ e e)
     number))
(define-language L2
  (e ::=
     (if e e e)
```

```
      true
      false))
(define-union-language L1-plus-L2 L1 L2)
```

is equivalent to this one:

```
(define-language L1-plus-L2
  (e ::=
     (+ e e)
     number
     (if e e e)
     true
     false))
```

If a language has a prefix, then all of the non-terminals from that language have the corresponding prefix in the union language. The prefix helps avoid unintended collisions between the constituent language's non-terminals.

For example, with two these two languages:

```
(define-language UT
  (e (e e)
     (λ (x) e)
     x))

(define-language WT
  (e (e e)
     (λ (x t) e)
     x)
  (t (→ t t)
     num))
```

then this declaration:

```
(define-union-language B (ut. UT) (wt. WT))
```

will create a language named `B` containing the non-terminals `ut.e`, `wt.e`, and `wt.t` consisting of the productions listed in the original languages.

```
(make-immutable-binding-hash lang [assocs]) → dict?
  lang : compiled-lang?
  assocs : (listof pair?) = '()
```

Returns an immutable dictionary where `alpha-equivalent?` keys are treated as the same.

Added in version 1.14 of package redex-lib.

```
(make-binding-hash lang [assocs]) → dict?
  lang : compiled-lang?
  assocs : (listof pair?) = '()
```

Returns a mutable dictionary where alpha-equivalent? keys are treated as the same.

Added in version 1.14 of package redex-lib.

```
(language-nts lang) → (listof symbol?)
  lang : compiled-lang?
```

Returns the list of non-terminals (as symbols) that are defined by this language.

```
(compiled-lang? l) → boolean?
  l : any/c
```

Returns #t if its argument was produced by language, #f otherwise.

```
(default-language) → (or/c false/c compiled-lang?)
(default-language lang) → void?
  lang : (or/c false/c compiled-lang?)
```

The value of this parameter is used by the default value of (default-equiv) to determine what language to calculate alpha-equivalence in. By default, it is #f, which acts as if it were a language with no binding forms. In that case, alpha-equivalence is the same thing as equal?.

The default-language parameter is set to the appropriate language inside judgment forms and metafunctions, and by apply-reduction-relation.

```
(alpha-equivalent? lang lhs rhs) → boolean?
  lang : compiled-lang?
  lhs : any/c
  rhs : any/c
(alpha-equivalent? lhs rhs) → boolean?
  lhs : any/c
  rhs : any/c
```

Returns #t if (according to the binding specification in *lang*) the bound names in *lhs* and *rhs* have the same structure and, in everything but bound names, they are equal?. If *lang* has no binding forms, terms have no bound names and therefore alpha-equivalent? is the same as equal?.

If the *lang* argument is not supplied, it defaults to the value of (default-language), which must not #f.

```
(substitute val old-var new-val)
(substitute val (old-var new-val) ...)
```

A metafunction that returns a value like *val*, except that any free occurences of *old-var* have been replaced with *new-val*, in a capture-avoiding fashion. The bound names of *val* may be freshened in order to accomplish this, based on the binding information in (default-language) (this is unlike normal metafunctions, which are defined in a particular language).

If a list of susbtitutions is provided, they will be applied simultaneously.

```
> (define-language lc-bind
    (e ::= (e e)
           x
           (λ (x ...) e))
    (x ::= variable-not-otherwise-mentioned)
    #:binding-forms
    (λ (x ...) e #:refers-to (shadow x ...)))
> (define-metafunction lc-bind
    β-reduce : (λ (x ..._1) e) e ..._1 -> e
    [(β-reduce (λ (x ...) e) e_x ...) (substitute e [x e_x] ...)])
> (term (β-reduce (λ (x y) (x y)) y z))
'(y z)
```

Note that `substitute` is merely a convenience metafunction. Any manually-written substitution in the correct language will also be capture-avoiding, provided that the language's binding forms are correctly defined. However, `substitute` may be significantly faster.

Changed in version 1.19 of package `redex-lib`: Added support for simultaneous substitutions

## 4.4 Reduction Relations

```
(reduction-relation language domain codomain base-arrow
                    reduction-case ...
                    shortcuts)
```

```
        domain =
               | #:domain pattern

      codomain =
               | #:codomain pattern

    base-arrow =
               | #:arrow base-arrow-name

reduction-case = (arrow-name pattern term red-extras ...)

    red-extras = rule-name
               | (fresh fresh-clause ...)
               | (side-condition racket-expression)
               | (where pattern term)
               | (where/hidden pattern term)
               | (where/error pattern term)
               | (bind pattern term)
               | (bind/hidden pattern term)
               | (judgment-holds (judgment-form-id pat/term ...))
               | (judgment-holds (relation-id term ...))
               | (side-condition/hidden racket-expression)

     shortcuts =
               | with shortcut ...

      shortcut = [(old-arrow-name pattern term)
                  (new-arrow-name identifier identifier)]

     rule-name = identifier
               | string
               | (computed-name racket-expression)

  fresh-clause = var
               | ((var1 ...) (var2 ...))

      pat/term = pattern
               | term
```

Defines a reduction relation case-wise, one case for each of the *reduction-case* clauses.

The optional *domain* and *codomain* clauses provide contracts for the relation. If the *codomain* is not present, but the *domain* is, then the codomain is expected to be the same as the domain.

The *arrow-name* in each *reduction-case* clause is either *base-arrow-name* (default

-->) or an arrow name defined by *shortcuts* (described below). In either case, the pattern refers to *language* and binds variables in the corresponding term. Following the pattern and term can be the name of the reduction rule and declarations of fresh variables and side-conditions.

For example, the expression

```
(reduction-relation
 lc-lang
 (--> (in-hole c_1 ((λ (variable_i ...) e_body) v_i ...))
      (in-hole c_1 ,(foldl lc-subst
                           (term e_body)
                           (term (v_i ...))
                           (term (variable_i ...))))
      beta-v))
```

defines a reduction relation for the *lc-lang* grammar.

A rule's name (used in typesetting, the `stepper`, `traces`, and `apply-reduction-relation/tag-with-names`) can be given as a literal (an identifier or a string) or as an expression that computes a name using the values of the bound pattern variables (much like the rule's right-hand side). Some operations require literal names, so a rule definition may provide both a literal name and a computed name. In particular, only rules that include a literal name may be replaced using `extend-reduction-relation`, used as breakpoints in the `stepper`, and selected using `render-reduction-relation-rules`. The output of `apply-reduction-relation/tag-with-names`, `traces`, and the `stepper` prefers the computed name, if it exists. Typesetting a rule with a computed name shows the expression that computes the name only when the rule has no literal name or when it would not typeset in pink due to `with-unquote-rewriters` in the context; otherwise, the literal name (or nothing) is shown.

Fresh variable clauses generate variables that do not occur in the term being reduced. If the *fresh-clause* is a variable, that variable is used both as a binding in the term and as the prefix for the freshly generated variable. (The variable does not have to be a non-terminal in the language of the reduction relation.)

The second form of *fresh-clause*s generates a sequence of variables. In that case, the ellipses are literal ellipses; that is, you must actually write ellipses in your rule. The variable *var1* is like the variable in first case of a *fresh-clause*; namely it is used to determine the prefix of the generated variables and it is bound in the right-hand side of the reduction rule, but unlike the single-variable fresh clause, it is bound to a sequence of variables. The variable *var2* is used to determine the number of variables generated and *var2* must be bound by the left-hand side of the rule.

The expressions within `side-condition` clauses and `side-condition/hidden` clauses are collected with `and` and used as guards on the case being matched. The argument to

each side-condition should be a Racket expression, and the pattern variables in the *pattern* are bound in that expression. A `side-condition/hidden` clause is the same as a `side-condition` clause, except that the condition is not rendered when typesetting via `redex/pict`.

Each *`where` clause* acts as a side condition requiring a successful pattern match, and it can bind pattern variables in the side-conditions (and `where` clauses) that follow and in the metafunction result.

A `where/hidden` clause is the same as a `where` clause, but the clause is not rendered when typesetting via `redex/pict`.

The `where/error` clause clause is like `where`, except that a failure to match is an error and, if multiple matches are possible, the right-hand side must produce the same result for each of the different matches (in the sense of `alpha-equivalent?` using the language that this reduction relation is defined with).

Each `judgment-holds` clause acts like a `where` clause, where the left-hand side pattern incorporates each of the patterns used in the judgment form's output positions.

Each *`shortcut`* clause defines arrow names in terms of *`base-arrow-name`* and earlier *`shortcut`* definitions. The left- and right-hand sides of a *`shortcut`* definition are identifiers, not patterns and terms. These identifiers need not correspond to non-terminals in *`language`* and if they do, that correspondence is ignored (more precisely, the shortcut is *not* restricted only to terms matching the non-terminal).

For example, this expression

```
(reduction-relation
 lc-num-lang
 (==> ((λ (variable_i ...) e_body) v_i ...)
      ,(foldl lc-subst
              (term e_body)
              (term (v_i ...))
              (term (variable_i ...))))
 (==> (+ number_1 ...)
      ,(apply + (term (number_1 ...)))))

 with
 [(--> (in-hole c_1 a) (in-hole c_1 b))
  (==> a b)])
```

defines reductions for the $\lambda$-calculus with numbers, where the `==>` shortcut is defined by reducing in the context `c`.

A `fresh` clause in *`reduction-case`* defined by shortcut refers to the entire term, not just the portion matched by the left-hand side of shortcut's use.

Changed in version 1.14 of package `redex-lib`: Added the `#:codomain` clause.

```
(extend-reduction-relation reduction-relation language more ...)
```

This form extends the reduction relation in its first argument with the rules specified in *more*. They should have the same shape as the rules (including the `with` clause) in an ordinary *reduction-relation*.

If the original reduction-relation has a rule with the same name as one of the rules specified in the extension, the old rule is removed.

In addition to adding the rules specified to the existing relation, this form also reinterprets the rules in the original reduction, using the new language.

```
(union-reduction-relations r ...) → reduction-relation?
  r : reduction-relation?
```

Combines all of the argument reduction relations into a single reduction relation that steps when any of the arguments would have stepped.

```
(reduction-relation->rule-names r) → (listof symbol?)
  r : reduction-relation?
```

Returns the names of the reduction relation's named clauses.

Added in version 1.20 of package `redex-lib`.

```
(compatible-closure reduction-relation lang non-terminal)
```

This accepts a reduction, a language, the name of a non-terminal in the language and returns the compatible closure of the reduction for the specified non-terminal.

In the below example, `addition-without-context` contains two rules that calculate sums. But, since it requires a `+` at the top level, it will not reduce subexpressions within a larger non-matching expression.

```
(define-language unary-arith
  [e ::= Z (S e) (+ e e)])

(define addition-without-context
  (reduction-relation
   unary-arith
   #:domain e
   (--> (+ (S e_1) e_2)
        (+ e_1 (S e_2)))
   (--> (+ Z e)
        e)))
```

```
(define nested-example (term (+ (+ (S Z) (S Z))
                                (+ (S Z) (S Z)))))

> (apply-reduction-relation addition-without-context
                            nested-example)
'()
```

The `compatible-closure` operator allows us to close `addition-without-context` over all nested `e` contexts and then we can use it to find nested sums.

```
(define addition-compat-closure
  (compatible-closure addition-without-context unary-arith e))

> (apply-reduction-relation* addition-compat-closure
                             nested-example)
'((S (S (S (S Z)))))
```

```
(context-closure reduction-relation lang pattern)
```

This accepts a reduction, a language, a pattern representing a context (i.e., that can be used as the first argument to `in-hole`; often just a non-terminal) in the language and returns the closure of the reduction in that context.

Continuing the example in the documentation for `compatible-closure`, one might find that there are too many reductions that can take place. The original example, in fact, reduces to two different terms in a single step.

```
> (apply-reduction-relation addition-compat-closure
                            nested-example)
'((+ (+ (S Z) (S Z)) (+ Z (S (S Z)))) (+ (+ Z (S (S Z))) (+ (S Z)
(S Z))))
```

If we wanted to force an order of evaluation, requiring that we evaluate the left side of the addition before moving on to the right, we can do that by limiting the context where the addition is performed. Here is one definition of a context that does that limitation.

```
(define-extended-language unary-arith-E unary-arith
  (E ::= hole (+ n E) (+ E e))
  (n ::= Z (S n)))
```

Now we can use the more general `compatible-closure` to close only over the places where *E* allows us to reduce

```
> (apply-reduction-relation
   (context-closure addition-without-context
                    unary-arith-E
                    E)
   nested-example)
'((+ (+ Z (S (S Z))) (+ (S Z) (S Z))))
```

```
(reduction-relation? v) → boolean?
  v : any/c
```

Returns `#t` if its argument is a reduction-relation, and `#f` otherwise.

```
(apply-reduction-relation r t) → (listof any/c)
  r : (or/c reduction-relation? IO-judgment-form?)
  t : any/c
```

This accepts reduction relation, a term, and returns a list of terms that the term reduces to.

```
(apply-reduction-relation/tag-with-names r
                                         t)
 → (listof (list/c (or/c #f string?) any/c))
  r : (or/c reduction-relation? IO-judgment-form?)
  t : any/c
```

Like `apply-reduction-relation`, but the result indicates the names of the reductions that were used.

```
 (apply-reduction-relation*
  r
  t
  [#:all? all
   #:cache-all? cache-all?
   #:stop-when stop-when]
   #:error-on-multiple? error-on-multiple?)
 → (listof any/c)
  r : (or/c reduction-relation? IO-judgment-form?)
  t : any/c
  all : boolean? = #f
  cache-all? : boolean? = (or all? (current-cache-all?))
  stop-when : (-> any/c any) = (λ (x) #f)
  error-on-multiple? : boolean?
```

Accepts a reduction relation and a term. Starting from `t`, it follows every reduction path and returns either all of the terms that do not reduce further or all of the terms, if `all?` is `#t`. If

there are infinite reduction sequences that do not repeat, this function will not terminate (it does terminate if the only infinite reduction paths are cyclic).

If the *cache-all?* argument is #t, then apply-reduction-relation* keeps a cache of all visited terms when traversing the graph and does not revisit any of them. This cache can, in some cases, use a lot of memory, so it is off by default and the cycle checking happens by keeping track only of the current path it is traversing through the reduction graph.

The *stop-when* argument controls the stopping criterion. Specifically, it is called with each term that apply-reduction-relation* encounters. If it ever returns a true value (anything except #f), then apply-reduction-relation* considers the term to be irreducible (and so returns it and does not try to reduce it further).

If *error-on-multiple?* is #t, then an error is signalled if any of the terms reduce to more than one other term.

Changed in version 1.18 of package redex-lib: Added *error-on-multiple?*.

```
(current-cache-all?) → boolean?
(current-cache-all? cache-all?) → void?
  cache-all? : boolean?
```

Controls the behavior of apply-reduction-relation* and test-->>'s cycle checking. See apply-reduction-relation* for more details.

Examples:

```
> (define-language empty-lang)
> (define R
    (reduction-relation
     empty-lang
     (--> 0 1)
     (--> 0 2)
     (--> 2 3)
     (--> 3 3)))
> (apply-reduction-relation R 0)
'(2 1)
> (apply-reduction-relation* R 0)
'(1)
```

```
-->
```

Recognized specially within reduction-relation. A --> form is an error elsewhere.

```
fresh
```

Recognized specially within reduction-relation. A fresh form is an error elsewhere.

```
with
```

Recognized specially within `reduction-relation`. A `with` form is an error elsewhere.

## 4.5 Other Relations

```
(define-metafunction language
  metafunction-contract
  [(name pattern ...) term metafunction-extras ...]
  ...)

metafunction-contract =
                      | id : pattern-sequence ... -> range
                        maybe-pre-condition
                        maybe-post-condition

 maybe-pre-condition = #:pre term
                      |

maybe-post-condition = #:post term
                      |

               range = pattern
                     | pattern or range
                     | pattern ∨ range
                     | pattern ∪ range

 metafunction-extras = (side-condition racket-expression)
                     | (side-condition/hidden racket-expression)
                     | (where pat term)
                     | (where/hidden pat term)
                     | (where/error pat term)
                     | (judgment-holds
                        (judgment-form-id pat/term ...))
                     | (judgment-holds
                        (relation-id term ...))
                     | (clause-name name)
                     | or term
```

A *metafunction* is a function on terms. The `define-metafunction` form builds a metafunction according to the pattern and right-hand-side expressions. The first argument indicates the language used to resolve non-terminals in the pattern expressions. Each of the rhs-expressions is implicitly wrapped in *term*.

The contract, if present, is matched against every input to the metafunction and, if the match fails, an exception is raised. If a metavariable is repeated in a contract, it does not require the terms to be equal, unless there is an underscore subscript (i.e., the binding works like it does in `define-language`, not how it works in the patterns in the left-hand sides of the metafunction clauses).

If present, the term inside the *maybe-pre-condition* is evaluated after a successful match to the input pattern in the contract (with any variables from the input contract bound). If it returns `#f`, then the input contract is considered to not have matched and an error is also raised. When a metafunction returns, the expression in the *maybe-post-condition* is evaluated (if present), with any variables from the input or output contract bound.

The `side-condition`, `hidden-side-condition`, `where`, `where/hidden`, and `where/error` clauses behave as in the `reduction-relation` form.

The resulting metafunction raises an exception recognized by `exn:fail:redex?` if no clauses match or if one of the clauses matches multiple ways (and that leads to different results for the different matches).

The `side-condition` extra is evaluated after a successful match to the corresponding argument pattern. If it returns `#f`, the clause is considered not to have matched, and the next one is tried. The `side-condition/hidden` extra behaves the same, but is not typeset.

The `where` and `where/hidden` extra are like `side-condition` and `side-condition/hidden`, except the match guards the clause. The `where/error` extra is like `where`, except that the pattern must match.

The `judgment-holds` clause is like `side-condition` and `where`, except the given judgment or relation must hold for the clause to be taken.

The *clause-name* is used only when typesetting. See `metafunction-cases`.

The *or* clause is used to define piecewise conditional metafunctions. In particular, if any of the `where` or `side-condition` clauses fail, then evaluation continues after an *or* clause, treating the term that follows as the result (subject to any subsequent `where` clauses or `side-condition`s. This construction is equivalent to simply duplicating the left-hand side of the clause, once for each *or* expression, but signals to the typesetting library to use a large left curly brace to group the conditions in the *or*.

For example, here are two equivalent definitions of a `biggest` metafunction that typeset differently:

Examples:

```
> (define-metafunction lc-lang
    biggest : natural natural -> natural
    [(biggest natural_1 natural_2)
     natural_2
```

```
     (side-condition (< (term natural_1) (term natural_2)))]
    [(biggest natural_1 natural_2)
     natural_1])
> (render-metafunction biggest)
```
biggest⟦$natural_1$,$natural_2$⟧ = $natural_2$
 where (<$natural_1$$natural_2$)
biggest⟦$natural_1$,$natural_2$⟧ = $natural_1$
```
> (define-metafunction lc-lang
    biggest : natural natural -> natural
    [(biggest natural_1 natural_2)
     natural_2
     (side-condition (< (term natural_1) (term natural_2)))

     or

     natural_1])
> (render-metafunction biggest)
```
biggest⟦$natural_1$,$natural_2$⟧ = $\begin{cases} natural_2 \text{ where } (<natural_1 natural_2) \\ natural_1 \text{ otherwise} \end{cases}$

Note that metafunctions are assumed to always return the same results for the same inputs, and their results are cached, unless `caching-enabled?` is set to `#f`. Accordingly, if a metafunction is called with the same inputs twice, then its body is only evaluated a single time.

As an example, these metafunctions finds the free variables in an expression in the *lc-lang* above:

```
(define-metafunction lc-lang
  free-vars : e -> (x ...)
  [(free-vars (e_1 e_2 ...))
   (∪ (free-vars e_1) (free-vars e_2) ...)]
  [(free-vars x) (x)]
  [(free-vars (λ (x ...) e))
   (- (free-vars e) (x ...))])
```

The first argument to define-metafunction is the grammar (defined above). Following that are three cases, one for each variation of expressions (e in *lc-lang*). The free variables of an application are the free variables of each of the subterms; the free variables of a variable is just the variable itself, and the free variables of a $\lambda$ expression are the free variables of the body, minus the bound parameters.

Here are the helper metafunctions used above.

```
(define-metafunction lc-lang
```

```
  ∪ : (x ...) ... -> (x ...)
  [(∪ (x_1 ...) (x_2 ...) (x_3 ...) ...)
   (∪ (x_1 ... x_2 ...) (x_3 ...) ...)]
  [(∪ (x_1 ...))
   (x_1 ...)]
  [(∪) ()])

(define-metafunction lc-lang
  - : (x ...) (x ...) -> (x ...)
  [(- (x ...) ()) (x ...)]
  [(- (x_1 ... x_2 x_3 ...) (x_2 x_4 ...))
   (- (x_1 ... x_3 ...) (x_2 x_4 ...))
   (side-condition (not (memq (term x_2) (term (x_3 ...)))))]
  [(- (x_1 ...) (x_2 x_3 ...))
   (- (x_1 ...) (x_3 ...))])
```

Note the side-condition in the second case of `-`. It ensures that there is a unique match for that case. Without it, `(term (- (x x) x))` would lead to an ambiguous match.

Changed in version 1.4 of package `redex-lib`: Added `#:post` conditions.
Changed in version 1.5: Added `or` clauses.

```
(define-metafunction/extension f language
  metafunction-contract
  [(g pattern ...) term metafunction-extras ...]
  ...)
```

Defines a metafunction *g* as an extension of an existing metafunction *f*. The metafunction *g* behaves as if *f*'s clauses were appended to its definition (with occurrences of *f* changed to *g* in the inherited clauses).

For example, `define-metafunction/extension` may be used to extend the free-vars function above to the forms introduced by the language *lc-num-lang*.

```
(define-metafunction/extension free-vars lc-num-lang
  free-vars-num : e -> (x ...)
  [(free-vars-num number)
   ()]
  [(free-vars-num (+ e_1 e_2))
   (∪ (free-vars-num e_1)
      (free-vars-num e_2))])
```

```
(in-domain? (metafunction-name term ...))
```

Returns `#t` if the inputs specified to *metafunction-name* are legitimate inputs according to *metafunction-name*'s contract, and `#f` otherwise.

```
(define-judgment-form language
  mode-spec
  contract-spec
  invariant-spec
  rule rule ...)
```

```
    mode-spec =
              | #:mode (form-id pos-use ...)

 contract-spec =
              | #:contract (form-id pattern-sequence ...)

invariant-spec =
              | #:inv term

      pos-use = I
              | O

         rule = [premise
                 ...
                 dashes rule-name
                 conclusion]
              | [conclusion
                 premise
                 ...
                 rule-name]

   conclusion = (form-id pat/term ...)

      premise = (judgment-form-id pat/term ...) maybe-ellipsis
              | (relation-id pat/term ...) maybe-ellipsis
              | (where pattern term)
              | (where/hidden pattern term)
              | (where/error pattern term)
              | (side-condition term)
              | (side-condition/hidden term)

    rule-name =
              | string
              | non-ellipsis-non-dashes-var

     pat/term = pattern
              | term

maybe-ellipsis =
              | ...

       dashes = ---
              | ----
              | -----
              | etc.
```

Defines *form-id* as a relation on terms via a set of inference rules.

If a *mode-spec* appears, each rule must be such that its premises can be evaluated left-to-right without "guessing" values for any of their pattern variables. Redex checks this property using *mode-spec* declaration, which partitions positions into inputs `I` and outputs `O`. Output positions in conclusions and input positions in premises must be *term*s; input positions in conclusions and output positions in premises must be *pattern*s. The *rule-name*s are used by `build-derivations` and by `render-judgment-form`.

If a *mode-spec* is not present, Redex cannot compute a derivation for the judgment form, instead it can check that a given derivation is valid according to the rules.

When the optional *contract-spec* declaration is present, Redex dynamically checks that the terms flowing through these positions match the provided patterns, raising an exception recognized by `exn:fail:redex?` if not. The term in the optional *invariant-spec* is evaluated after the output positions have been computed and the contract has matched successfully, with variables (that have underscores) from the contract bound; a result of `#f` is considered to be a contract violation and an exception is raised.

For example, the following defines addition on natural numbers:

```
> (define-language nats
    (n ::= z (s n)))
> (define-judgment-form nats
    #:mode (sum I I O)
    #:contract (sum n n n)
    [-----------  "zero"
     (sum z n n)]

    [(sum n_1 n_2 n_3)
     ------------------------ "add1"
     (sum (s n_1) n_2 (s n_3))])
```

When a judgment form has a mode, the `judgment-holds` form checks whether a judgment form holds for any assignment of pattern variables in output positions.

Examples:

```
> (judgment-holds (sum (s (s z)) (s z) (s (s (s z)))))
#t
> (judgment-holds (sum (s (s z)) (s z) (s (s (s n)))))
#t
> (judgment-holds (sum (s (s z)) (s z) (s (s (s (s n))))))
#f
```

Alternatively, this form constructs a list of terms based on the satisfying pattern variable assignments.

Examples:

```
> (judgment-holds (sum (s (s z)) (s z) (s (s (s n)))) n)
'(z)
> (judgment-holds (sum (s (s z)) (s z) (s (s (s (s n))))) n)
'()
> (judgment-holds (sum (s (s z)) (s z) (s (s (s n)))) (s n))
'((s z))
```

Declaring different modes for the same inference rules enables different forms of computation. For example, the following mode allows `judgment-holds` to compute all pairs with a given sum.

```
> (define-judgment-form nats
    #:mode (sumr O O I)
    #:contract (sumr n n n)
    [------------ "z"
     (sumr z n n)]

    [(sumr n_1 n_2 n_3)
     -------------------------- "s"
     (sumr (s n_1) n_2 (s n_3))])
> (judgment-holds (sumr n_1 n_2 (s (s z))) (n_1 n_2))
'(((s (s z)) z) ((s z) (s z)) (z (s (s z))))
```

In some situations, there is no mode that could be specified that Redex accepts. It is possible to leave off the mode in that case, as in this judgment form:

```
> (define-extended-language nat-exprs nats
    (e ::= (+ e e) n))
> (define-judgment-form nat-exprs
    #:contract (same-exp e e)

    [(sum n_1 n_2 n_3)
     ------------------------- "add"
     (same-exp (+ n_1 n_2) n_3)]

    [-------------- "refl"
     (same-exp e e)]

    [(same-exp e_1 e_2) (same-exp e_2 e_3)
     ----------------- "trans"
     (same-exp e_1 e_3)]

    [(same-exp e_2 e_1)
```

```
 ------------------ "sym"
 (same-exp e_1 e_2)]

[(same-exp e_1 e_2)
 ---------------------------------- "compat-l"
 (same-exp (+ e_1 e_3) (+ e_2 e_3))]

[(same-exp e_1 e_2)
 ---------------------------------- "compat-r"
 (same-exp (+ e_3 e_1) (+ e_3 e_2))])
```

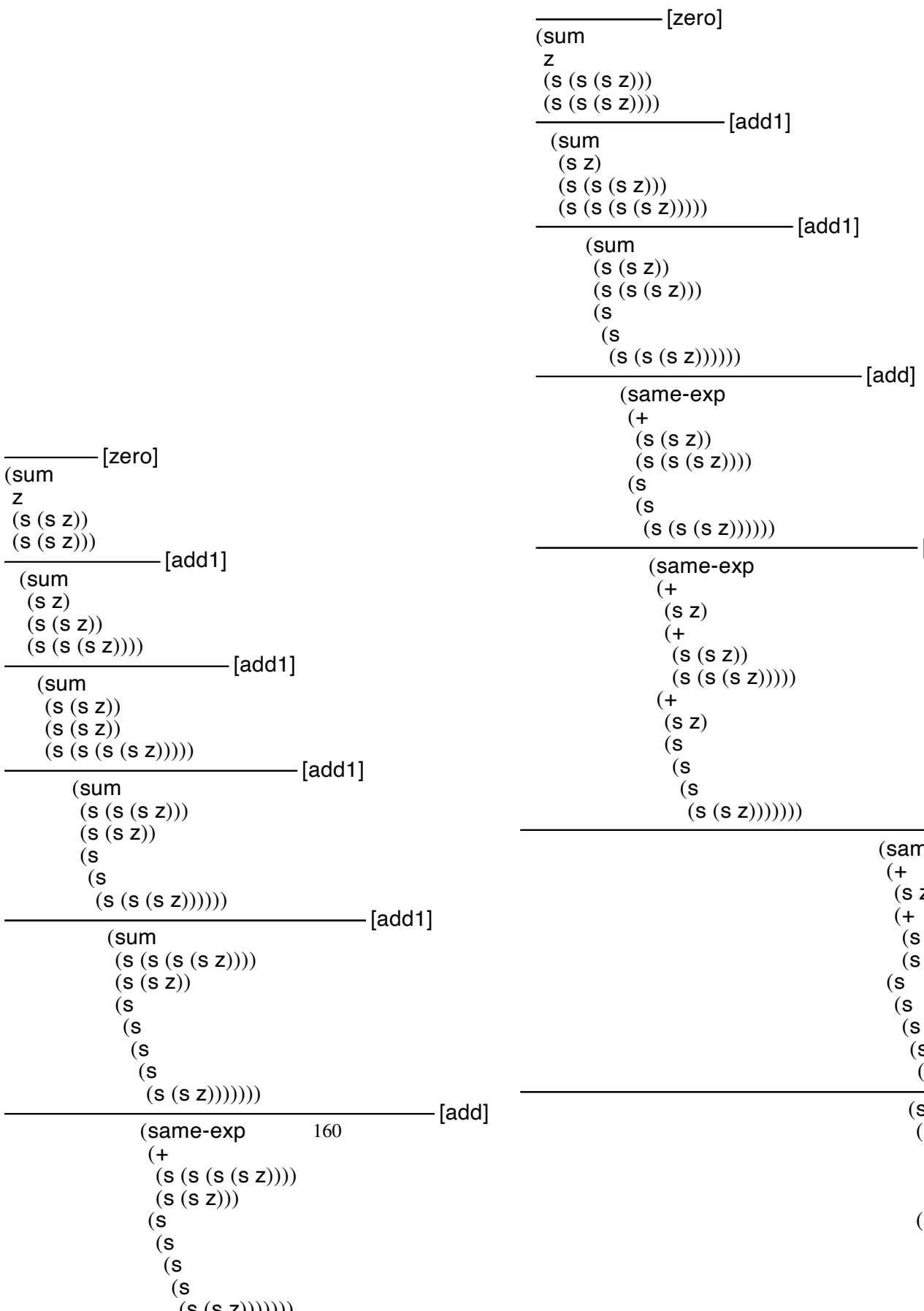
With a modeless judgment form, Redex cannot compute the entire derivation for you, but it can check that a given derivation is valid according to the rules in the judgment form. Here is one such derivation:

```
> (define same-exp-derivation
    (let* ([one `(s z)]
           [two `(s ,one)]
           [three `(s ,two)]
           [four `(s ,three)]
           [five `(s ,four)]
           [six `(s ,five)])
      (derivation
       `(same-exp (+ ,four ,two)
                  (+ ,one (+ ,two ,three)))
       "trans"
       (list
        (derivation `(same-exp (+ ,four ,two) ,six)
                    "add"
                    (list (car (build-derivations (sum ,four ,two n)))))
        (derivation
         `(same-exp ,six
                    (+ ,one (+ ,two ,three)))
         "sym"
         (list
          (derivation
           `(same-exp (+ ,one (+ ,two ,three))
                      ,six)
           "trans"
           (list
            (derivation
             `(same-exp (+ ,one (+ ,two ,three))
                        (+ ,one ,five))
             "compat-r"
             (list
              (derivation `(same-exp (+ ,two ,three)
```

```
                                        ,five)
                          "add"
                          (list (car (build-
derivations (sum ,two ,three n)))))))
            (derivation `(same-exp (+ ,one ,five)
                                   ,six)
                        "add"
                        (list (car (build-
derivations (sum ,one ,five n))))))))))))))
```

It is a bit hard to read in that form; here it is in a more traditional tree rendering:

```
> (parameterize ([pretty-print-columns 20])
    (derivation->pict nat-exprs same-exp-derivation))
```


[zero]
(sum
z
(s (s z))
(s (s z)))
[add1]
(sum
(s z)
(s (s z))
(s (s (s z))))
[add1]
(sum
(s (s z))
(s (s z))
(s (s (s (s z)))))
[add1]
(sum
(s (s (s z)))
(s (s z))
(s
(s
(s (s (s z))))))
[add1]
(sum
(s (s (s (s z))))
(s (s z))
(s
(s
(s
(s
(s (s z)))))))
[add]
(same-exp
(+
(s (s (s (s z))))
(s (s z)))
(s
(s
(s
(s
(s (s (s z))))))))
[zero]
(sum
z
(s (s (s z)))
(s (s (s z))))
[add1]
(sum
(s z)
(s (s (s z)))
(s (s (s (s z)))))
[add1]
(sum
(s (s z))
(s (s (s z)))
(s
(s
(s (s (s z))))))
[add]
(same-exp
(+
(s (s z))
(s (s (s z))))
(s
(s
(s (s (s z))))))
(same-exp
(+
(s z)
(+
(s (s z))
(s (s (s z)))))
(+
(s z)
(s
(s
(s
(s (s z)))))))
(sam
(+
(s z
(+
(s
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(s

And using `judgment-holds`, we see that Redex agrees it is a valid derivation for `same-exp`.

```
> (judgment-holds same-exp same-exp-derivation)
#t
```

The premises must be in the same order in the `derivation` struct's *`subs`* field as they appear in the definition of the judgment form.

A rule's `where`, `where/hidden`, and `where/error` premises behave as in `reduction-relation` and `define-metafunction`.

Examples:

```
> (define-judgment-form nats
    #:mode (le I I)
    #:contract (le n n)
    [--------
     (le z n)]

    [(le n_1 n_2)
     --------------------
     (le (s n_1) (s n_2))])
> (define-metafunction nats
    pred : n -> n or #f
    [(pred z) #f]
    [(pred (s n)) n])
> (define-judgment-form nats
    #:mode (gt I I)
    #:contract (gt n n)
    [(where n_3 (pred n_1))
     (le n_2 n_3)
     ----------------------
     (gt n_1 n_2)])
> (judgment-holds (gt (s (s z)) (s z)))
#t
> (judgment-holds (gt (s z) (s z)))
#f
```

A rule's `side-condition` and `side-condition/hidden` premises are similar to those in `reduction-relation` and `define-metafunction`, except that they do not implicitly unquote their right-hand sides. In other words, a premise of the form `(side-condition term)` is close to the premise `(where #t term)`, except it does not typeset with the "#t = ", as that would and it holds whenever the expression evaluates to any non `#f` value (not just `#t`).

Judgments with exclusively `I` mode positions may also be used in *`term`*s in a manner similar to metafunctions, and evaluate to a boolean.

Examples:

```
> (term (le (s z) (s (s z))))
#t
> (term (le (s z) z))
#f
```

A literal ellipsis may follow a judgment premise when a template in one of the judgment's input positions contains a pattern variable bound at ellipsis-depth one.

Examples:

```
> (define-judgment-form nats
    #:mode (even I)
    #:contract (even n)

    [-------- "evenz"
     (even z)]

    [(even n)
     ---------------- "even2"
     (even (s (s n)))])
> (define-judgment-form nats
    #:mode (all-even I)
    #:contract (all-even (n ...))
    [(even n) ...
     ------------------
     (all-even (n ...))])
> (judgment-holds (all-even (z (s (s z)) z)))
#t
> (judgment-holds (all-even (z (s (s z)) (s z))))
#f
```

Redex evaluates premises depth-first, even when it doing so leads to non-termination. For example, consider the following definitions:

```
> (define-language vertices
    (v a b c))
> (define-judgment-form vertices
    #:mode (edge I O)
    #:contract (edge v v)
    [(edge a b)]
    [(edge b c)])
> (define-judgment-form vertices
    #:mode (path I I)
    #:contract (path v v)
```

```
[----------
 (path v v)]

[(path v_2 v_1)
 --------------
 (path v_1 v_2)]

[(edge v_1 v_2)
 (path v_2 v_3)
 --------------
 (path v_1 v_3)])
```

Due to the second `path` rule, the follow query fails to terminate:

```
> (judgment-holds (path a c))
```

There are three example files that come with Redex that demonstrates three use cases.

- `"typing-rules.rkt"` — defines a type system in a way that supports mechanized typesetting. When a typing judgment form can be given a mode, it can also be encoded as a metafunction using `where` clauses as premises, but Redex cannot typeset that encoding as inference rules.
- `"sos.rkt"` — defines an SOS-style semantics in a way that supports mechanized typesetting.
- `"multi-val.rkt"` — defines a judgment form that serves as a multi-valued metafunction.

These files can be found via DrRacket's File|Open Require Path... menu item. Type `redex/examples/d/` into the dialog and then choose one of the names listed above. Or, evaluate the expression

```
(collection-file-path «filename.rkt»
                      "redex"
                      "examples"
                      "define-judgment-form")
```

replacing **«filename.rkt»** with one of the names listed above.

Note that `current-traced-metafunctions` also traces judgment forms and is helpful when debugging.

```
(define-extended-judgment-form language judgment-form-id
  mode-spec
  contract-spec
  invariant-spec
  rule ...)
```

Defines a new judgment form that extends *judgment-form-id* with additional rules. The *mode-spec*, *contract-spec*, *invariant-spec*, and *rule*s are as in `define-judgment-form`.

The mode specification in this judgment form and the original must be the same.

```
(define-overriding-judgment-form language judgment-form-id
  mode-spec
  contract-spec
  invariant-spec
  rule ...)
```

Defines a new judgment form that extends *judgment-form-id* with additional rules, replacing rules from *judgment-form-id* that have the same name as any of the new rules.

The *mode-spec*, *contract-spec*, *invariant-spec*, and *rule*s are as in `define-judgment-form`.

The mode specification in this judgment form and the original must be the same.

```
(judgment-holds judgment-or-relation)
(judgment-holds judgment-or-relation term)
(judgment-holds judgment-form-id derivation-expr)

judgment-or-relation = (judgment-form-id pat/term ...)
                     | (relation-id pat/term ...)
```

In its first form, checks whether *judgment-or-relation* holds for any assignment of the pattern variables in *judgment-form-id*'s output positions (or just that it holds in the case that a relation from `define-relation` is used).

In its second form, produces a list of terms by instantiating the supplied term template with each satisfying assignment of pattern variables. In the second case, if a relation is supplied, there are no pattern variables, so the result is either a list with one element or the empty list.

In both of the first two forms, any given judgment form must have a mode.

In its third form, the *judgment-form-id* must not have a mode, and the *derivation-expr* must produce a `derivation` struct. The result of `judgment-holds` is `#t` when the

derivation is valid, according to the rules of the judgment form, and `#f` otherwise. Note that the premises of the derivation must appear in the same order as the premises in the definition of the judgment form.

```
> (judgment-holds (sum (s (s z)) (s z) n))
#t
> (judgment-holds (sum (s (s z)) (s z) n) n)
'((s (s (s z))))
```

See `define-judgment-form` for more examples.

```
(judgment-form->rule-names
 r
 [#:include-unnamed? include-unnamed?])
→ (listof (or/c symbol? #f))
 r : judgment-form?
 include-unnamed? : any/c = #f
```

Returns the names of the judgment form's clauses.

If *include-unnamed?* is `#f` (the default) then any case that does not have a name is not included in the result and the result list contains only `symbol?`s.

Otherwise, each case is included in the result and unnamed cases have a `#false` at that position in the result list.

```
(build-derivations judgment-or-relation)
```

Constructs all of the `derivation` trees for *judgment-or-relation*.

Example:

```
> (build-derivations (even (s (s z))))
(list
 (derivation
  '(even (s (s z)))
  "even2"
  (list (derivation '(even z) "evenz" '()))))
```

```
(struct derivation (term name subs)
    #:extra-constructor-name make-derivation)
  term : any/c
  name : (or/c string? #f)
  subs : (listof derivation?)
```

Represents a derivation from a judgment form.

The `term` field holds an s-expression based rendering of the conclusion of the derivation, the `name` field holds the name of the clause with `term` as the conclusion, and `subs` contains the sub-derivations.

See also `build-derivations`.

```
I
```

Recognized specially within `define-judgment-form`, the `I` keyword is an error elsewhere.

```
O
```

Recognized specially within `define-judgment-form`, the `O` keyword is an error elsewhere.

```
(define-relation language
  relation-contract
  [(name pattern ...)
   term ...
   metafunction-extras ...] ...)

relation-contract =
                  | form-id ⊂ pattern x ... x pattern
                  | form-id ⊆ pattern × ... × pattern
```

Similar to `define-judgment-form` but suitable only when every position is an input. Querying the result uses `judgment-holds` or the same syntax as metafunction application.

The contract specification for a relation restricts the patterns that can be used as input to a relation. For each argument to the relation, there should be a single pattern, using `x` or `×` to separate the argument contracts.

Examples:

```
> (define-language types
    ((τ σ) int
           num
           (τ → τ)))
> (define-relation types
    subtype ⊆ τ × τ
    [(subtype int num)]
    [(subtype (τ_1 → τ_2) (σ_1 → σ_2))
     (subtype σ_1 τ_1)
     (subtype τ_2 σ_2)]
    [(subtype τ τ)])
```

```
> (judgment-holds (subtype int num))
#t
> (judgment-holds (subtype (int → int) (num → num)))
#f
> (judgment-holds (subtype (num → int) (num → num)))
#t
```

```
(judgment-form? v) → boolean?
  v : any/c
```

Identifies values bound to identifiers introduced by `define-judgment-form` and `define-relation`.

```
(IO-judgment-form? v) → boolean?
  v : any/c
```

Identifies values bound to identifiers introduced by `define-judgment-form` when the mode is `(I O)` or `(O I)`.

```
(current-traced-metafunctions) → (or/c 'all (listof symbol?))
(current-traced-metafunctions traced-metafunctions) → void?
  traced-metafunctions : (or/c 'all (listof symbol?))
```

Controls which metafunctions and judgment forms are currently being traced. If it is `'all`, all of them are. Otherwise, the elements of the list name the metafunctions and judgments to trace.

The tracing looks just like the tracing done by the `racket/trace` library, except that the first column printed by each traced call indicate if this call to the metafunction is cached. Specifically, a `c` is printed in the first column if the result is just returned from the cache and a space is printed if the metafunction or judgment call is actually performed.

Defaults to `'()`.

Examples:

```
> (define-judgment-form nats
    #:mode (odd I)
    #:contract (odd n)

    [-------- "oddsz"
     (odd (s z))]

    [(odd n)
     ---------------- "odd2"
     (odd (s (s n)))])
```

```
> (parameterize ([current-traced-metafunctions '(odd)])
    (judgment-holds (odd (s (s (s z))))))
 >(odd (s (s (s z))))
 > (odd (s z))
 < ((odd (s z)))
 <((odd (s (s (s z)))))
#t
> (parameterize ([current-traced-metafunctions '(odd)])
    (judgment-holds (odd (s (s (s (s (s z))))))))
 >(odd (s (s (s (s (s z))))))
c> (odd (s (s (s z))))
 < ((odd (s (s (s z)))))
 <((odd (s (s (s (s (s z)))))))
#t
```

## 4.6 Testing

```
(test-equal e1 e2 option)

option = #:equiv pred-expr
       |

  pred-expr : (-> any/c any/c any/c)
```

Tests to see if *e1* is equal to *e2*, using *pred-expr* as the comparison. It defaults to `(default-equiv)`.

Examples:

```
> (define-language L
    (bt ::=
        empty
        (node any bt bt))
    (lt ::=
        empty
        (node any lt empty)))
> (define-metafunction L
    linearize/a : bt lt -> lt
    [(linearize/a empty lt) lt]
    [(linearize/a (node any_val bt_left bt_right) lt)
     (node any_val (linearize/a bt_left (linearize/a bt_right lt)) empty)])
> (define-metafunction L
    linearize : bt -> lt
    [(linearize bt) (linearize/a bt empty)])
```

```
> (test-equal (term (linearize empty))
              (term empty))
> (test-equal (term (linearize (node 1
                                     (node 2 empty empty)
                                     (node 3 empty empty))))
              (term (node 1 (node 2 (node 3 empty empty) empty) empty)))
> (test-results)
Both tests passed.
```

```
(test-->> rel-expr option ... e1-expr e2-expr ...)

option = #:cycles-ok
       | #:equiv pred-expr
       | #:pred pred-expr

  rel-expr : (or/c reduction-relation? IO-judgment-form?)
  pred-expr : (-> any/c any)
  e1-expr : any/c
  e2-expr : any/c
```

Tests to see if the term *e1-expr*, reduces to the terms *e2-expr* under *rel-expr*, using *pred-expr* to determine equivalence.

If `#:pred` is specified, it is applied to each reachable term until one of the terms fails to satisfy the predicate (i.e., the predicate returns `#f`). If that happens, then the test fails and a message is printed with the term that failed to satisfy the predicate.

The procedure supplied after `#:equiv` is always passed the result of reducing the expression as its first argument and (one of) the expected result(s) as its second argument.

This test uses `apply-reduction-relation*`, so it does not terminate when the resulting reduction graph is infinite, although it does terminate if there are cycles in the (finite) graph.

If `#:cycles-ok` is not supplied then any cycles detected are treated as a test failure. If a *pred-expr* is supplied, then it is used to compare the expected and actual results. If it isn't supplied, then `(default-equiv)` is used.

```
(test--> rel-expr option ... e1-expr e2-expr ...)

option = #:equiv pred-expr

  rel-expr : (or/c reduction-relation? IO-judgment-form?)
  pred-expr : (-> any/c any/c any/c)
  e1-expr : any/c
  e2-expr : any/c
```

Tests to see if the term *e1-expr*, reduces to the terms *e2-expr* in a single *rel-expr* step, using *pred-expr* to determine equivalence (or `(default-equiv)` if *pred-expr* isn't specified).

Examples:

```
> (define-language L
    (i integer))
> (define R
    (reduction-relation
     L
     (--> i i)
     (--> i ,(add1 (term i)))))
> (define (mod2=? i j)
    (= (modulo i 2) (modulo j 2)))
> (test--> R #:equiv mod2=? 7 1)
FAILED :10.0
expected: 1
  actual: 8
  actual: 7
> (test--> R #:equiv mod2=? 7 1 0)
> (test-results)
1 test failed (out of 2 total).
```

```
(test-->>∃ option ... rel-expr start-expr goal-expr)

option = #:steps steps-expr

  rel-expr : (or/c reduction-relation? IO-judgment-form?)
  start-expr : any/c
  goal-expr : (or/c (-> any/c any/c)
                    (not/c procedure?))
  steps-expr : (or/c natural-number/c +inf.0)
```

Tests to see if the term *start-expr* reduces according to the reduction relation *rel-expr* to a term specified by *goal-expr* in *steps-expr* or fewer steps (default 1,000). The specification *goal-expr* may be either a predicate on terms or a term itself.

```
test-->>E
```

An alias for `test-->>∃`.

Examples:

```
> (define-language L
    (n natural))
> (define succ-mod8
    (reduction-relation
     L
     (--> n ,(modulo (add1 (term n)) 8))))
> (test-->>∃ succ-mod8 6 2)
> (test-->>∃ succ-mod8 6 even?)
> (test-->>∃ succ-mod8 6 8)
FAILED :17.0
term 8 not reachable from 6
> (test-->>∃ #:steps 6 succ-mod8 6 5)
FAILED :18.0
term 5 not reachable from 6 (within 6 steps)
> (test-results)
2 tests failed (out of 4 total).
```

```
(test-judgment-holds (judgment-form-or-relation pat/term ...))
(test-judgment-holds modeless-judgment-form derivation-expr)
```

In the first form, tests to see if `(judgment-form-or-relation pat/term ...)` holds. In the second form, tests to see if the result of *derivation-expr* is a derivation and, if so, that it is derivable using *modeless-judgment-form*.

```
(test-predicate p? e)
```

Tests to see if the value of *e* matches the predicate *p?*.

```
(test-match lang-id pat e)
```

Tests to see if the value of *e* matches, via `redex-match?`, the pattern *pat*.

Examples:

```
> (define-language L
    (n natural))
> (test-match L n (term 1))
> (test-match L n (term #t))
FAILED :22.0
  did not match pattern "n": #t
> (test-results)
1 test failed (out of 2 total).
```

```
(test-no-match lang-id pat e)
```

Tests to see if the value of `e` *does not match*, via `redex-match?`, the pattern `pat`.

Examples:

```
> (define-language L
    (n natural))
> (test-no-match L n (term 1))
FAILED :25.0
  did match pattern "n": 1
> (test-no-match L n (term #t))
> (test-results)
1 test failed (out of 2 total).
```

```
(test-results) → void?
```

Prints out how many tests passed and failed, and resets the counters so that next time this function is called, it prints the test results for the next round of tests.

```
(default-equiv) → (-> any/c any/c any/c)
(default-equiv equiv) → void?
  equiv : (-> any/c any/c any/c)
```

The value of this parameter is used as the default value of the equivalence predicates for `test-equal`, `test-->`, and `test-->>`.

It defaults to `(lambda (lhs rhs) (alpha-equivalent? (default-language) lhs rhs))`.

```
(make-coverage subject)

subject = metafunction
        | relation-expr
```

Constructs a structure (recognized by `coverage?`) to contain per-case test coverage of the supplied metafunction or reduction relation. Use with `relation-coverage` and `covered-cases`.

```
(coverage? v) → boolean?
  v : any/c
```

Returns `#t` for a value produced by `make-coverage` and `#f` for any other.

```
(relation-coverage) → (listof coverage?)
(relation-coverage tracked) → void?
  tracked : (listof coverage?)
```

Redex populates the coverage records in *tracked* (default null), counting the times that tests exercise each case of the associated metafunction and relations.

```
(covered-cases c) → (listof (cons/c string? natural-number/c))
  c : coverage?
```

Extracts the coverage information recorded in c, producing an association list mapping names (or source locations, in the case of metafunctions or unnamed reduction-relation cases) to application counts.

Examples:

```
> (define-language empty-lang)
> (define-metafunction empty-lang
    [(plus number_1 number_2)
     ,(+ (term number_1) (term number_2))])
> (define equals
    (reduction-relation
     empty-lang
     (--> (+) 0 "zero")
     (--> (+ number) number)
     (--> (+ number_1 number_2 number ...)
          (+ (plus number_1 number_2)
             number ...)
          "add")))
> (let ([equals-coverage (make-coverage equals)]
        [plus-coverage (make-coverage plus)])
    (parameterize ([relation-coverage (list equals-coverage
                                            plus-coverage)])
      (apply-reduction-relation* equals (term (+ 1 2 3)))
      (values (covered-cases equals-coverage)
              (covered-cases plus-coverage))))
'(("#f:30:0" . 1) ("add" . 2) ("zero" . 0))
'(("#f:29:0" . 2))
```

```
(generate-term from-pattern)
(generate-term from-judgment-form)
(generate-term from-metafunction)
(generate-term from-reduction-relation)
```

```
          from-pattern = language pattern size-expr kw-args ...
                       | language pattern
                       | language pattern #:i-th index-expr
                       | language pattern #:i-th

     from-judgment-form = language #:satisfying
                          (judgment-form-id pattern ...)
                       | language #:satisfying
                          (judgment-form-id pattern ...)
                          size-expr

      from-metafunction = language #:satisfying
                          (metafunction-id pattern ...) = pattern
                       | language #:satisfying
                          (metafunction-id pattern ...) = pattern
                          size-expr
                       | #:source metafunction size-expr kw-args
                       | #:source metafunction

from-reduction-relation = #:source reduction-relation-expr
                          size-expr kw-args ...
                       | #:source reduction-relation-expr

               kw-args = #:attempt-num attempt-num-expr
                       | #:retries retries-expr

  size-expr : natural-number/c
  attempt-num-expr : natural-number/c
  retries-expr : natural-number/c
```

Generates terms in a number of different ways:

- *from-pattern*: In the first case, randomly makes an expression matching the given pattern whose size is bounded by *size-expr*; the second returns a function that accepts a size bound and returns a random term. Calling this function (even with the same size bound) may be more efficient than using the first case.

  Examples:

```
> (define-language L
    (e ::=
       (e e)
       (λ (x) e)
       x)
    (x ::= a b c))
```

```
> (for/list ([i (in-range 10)])
    (generate-term L e 3))
'(((λ (b) (λ (b) b)) b)
  (b ((b b) (a a)))
  (((λ (a) b) c) c)
  (λ (a) ((λ (c) a) (b b)))
  (c b)
  c
  ((a a) ((a b) (c b)))
  c
  (λ (b) c)
  (λ (a) ((a a) (b a))))
```

The `#:i-th` option uses an enumeration of the non-terminals in a language. If *index-expr* is supplied, `generate-term` returns the corresponding term and if it isn't, `generate-term` returns a function from indices to terms.

Example:

```
> (for/list ([i (in-range 9)])
    (generate-term L e #:i-th i))
'(a (a a) (λ (a) a) b (a (a a)) (λ (b) a) c ((a a) a) (λ (c)
a))
```

Base type enumerations such as `boolean`, `natural` and `integer` are what you might expect:

Examples:

```
> (for/list ([i (in-range 10)])
    (generate-term L boolean #:i-th i))
'(#t #f #t #f #t #f #t #f #t #f)
> (for/list ([i (in-range 10)])
    (generate-term L natural #:i-th i))
'(0 1 2 3 4 5 6 7 8 9)
> (for/list ([i (in-range 10)])
    (generate-term L integer #:i-th i))
'(0 1 -1 2 -2 3 -3 4 -4 5)
```

The `real` base type enumeration consists of all integers and flonums, and the `number` pattern consists of complex numbers with real and imaginary parts taken from the `real` enumeration.

Examples:

```
> (for/list ([i (in-range 20)])
    (generate-term L real #:i-th i))
```

```
'(0
  +inf.0
  1
  -inf.0
  -1
  +nan.0
  2
  0.0
  -2
  5e-324
  3
  -5e-324
  -3
  1e-323
  4
  -1e-323
  -4
  1.5e-323
  5
  -1.5e-323)
> (for/list ([i (in-range 20)])
    (generate-term L number #:i-th i))
'(+inf.0
  0
  +inf.0+inf.0i
  1
  0.0+inf.0i
  -inf.0+inf.0i
  -inf.0
  0+1i
  +nan.0+inf.0i
  -1
  0.0-inf.0i
  0.0+inf.0i
  +nan.0
  0-1i
  5e-324+inf.0i
  2
  0.0+nan.0i
  -5e-324+inf.0i
  0.0
  0+2i)
```

The `string` enumeration produces all single character strings before going on to strings with multiple characters. For each character it starts the lowercase Latin characters, then uppercase Latin, and then every remaining Unicode character. The `vari-`

`able` enumeration is the same, except it produces symbols instead of strings.

Examples:

```
> (generate-term L string #:i-th 0)
"a"
> (generate-term L string #:i-th 1)
"b"
> (generate-term L string #:i-th 26)
"A"
> (generate-term L string #:i-th 27)
"B"
> (generate-term L string #:i-th 52)
"\u0000"
> (generate-term L string #:i-th 53)
"\u0001"
> (generate-term L string #:i-th 956)
"μ"
> (generate-term L variable #:i-th 1)
'b
> (generate-term L variable #:i-th 27)
'B
```

The `variable-prefix`, `variable-except`, and `variable-not-otherwise-mentioned` are defined similarly, as you expect.

Examples:

```
> (define-language L
    (used ::= a b c)
    (except ::= (variable-except a))
    (unused ::= variable-not-otherwise-mentioned))
> (for/list ([i (in-range 10)])
    (generate-term L (variable-prefix a:) #:i-th i))
'(a:a a:b a:c a:d a:e a:f a:g a:h a:i a:j)
> (for/list ([i (in-range 10)])
    (generate-term L except #:i-th i))
'(b c d e f g h i j k)
> (for/list ([i (in-range 10)])
    (generate-term L unused #:i-th i))
'(d e f g h i j k l m)
```

Finally, the `any` pattern enumerates terms of the above base types.

Example:

```
> (for/list ([i (in-range 20)])
    (generate-term L any #:i-th i))
```

```
'(()
  (())
  +inf.0
  (() ())
  "a"
  ((()))
  #t
  ((()) ())
  a
  (() . +inf.0)
  0
  ((()) . +inf.0)
  "b"
  (+inf.0)
  #f
  (+inf.0 ())
  b
  (+inf.0 . +inf.0)
  +inf.0+inf.0i
  (() () ()))
```

In addition, all other pattern types are supported except for mismatch repeat `..._!_` patterns and `side-condition` patterns.

The enumerators do not repeat terms unless the given pattern is ambiguous. Roughly speaking, the enumerator generates all possible ways that a pattern might be parsed and since ambiguous patterns have multiple ways they might be parsed, those multiple parsings turn into repeated elements in the enumeration.

Example:

```
> (for/list ([i (in-range 9)])
    (generate-term L (boolean_1 ... boolean_2 ...) #:i-th i))
'(() (#t) (#t) (#t #t) (#f) (#t #f) (#f) (#f #t) (#f #f))
```

Other sources of ambiguity are `in-hole` and overlapping non-terminals.

Examples:

```
> (define-language L
    (e ::= (e e) (λ (x) e) x)
    (E ::= hole (e E) (E e))
    (x ::= a b c))
> (for/list ([i (in-range 9)])
    (generate-term L (in-hole E e) #:i-th i))
'(a
  (a a)
  (a a)
```

```
  (a (a a))
  (λ (a) a)
  (a (λ (a) a))
  (a a)
  ((a a) a)
  ((λ (a) a) a))
> (define-language L
    (overlap ::= natural integer))
> (for/list ([i (in-range 10)])
    (generate-term L overlap #:i-th i))
'(0 0 1 1 2 -1 3 2 4 -2)
```

For similar reasons, enumerations for mismatch patterns (using `_!_`) do not work properly when given ambiguous patterns; they may repeat elements of the enumeration.

Examples:

```
> (define-language Bad
    (ambig ::= (x ... x ...)))
> (generate-term Bad (ambig_!_1 ambig_!_1) #:i-th 4)
'((x x) ())
```

In this case, the elements of the resulting list are the same, even though they should not be, according to the pattern. Internally, the enumerator has discovered two different ways to generate `ambig` (one where the `x` comes from the first ellipses and one from the second) but those two different ways produce the same term and so the enumerator incorrectly produces `(x x)`.

See also `redex-enum`.

- *`from-judgment-form`*: Randomly picks a term that satisfies the given use of the judgment form.

  Examples:

```
> (define-language L
    (nat ::= Z (S nat)))
> (define-judgment-form L
    #:mode (sum I I O)
    [---------------
     (sum Z nat nat)]
    [(sum nat_1 nat_2 nat_3)
     -------------------------------
     (sum (S nat_1) nat_2 (S nat_3))])
> (for/list ([i (in-range 10)])
    (generate-term L #:satisfying
                   (sum nat_1 nat_2 nat_3)
                   3))
```

```
'((sum (S (S (S (S Z)))) (S (S Z)) (S (S (S (S (S (S Z)))))))
  (sum Z (S Z) (S Z))
  (sum Z (S Z) (S Z))
  (sum (S (S (S Z))) Z (S (S (S Z))))
  (sum (S (S (S Z))) (S Z) (S (S (S (S Z)))))
  (sum Z Z Z)
  (sum (S (S Z)) Z (S (S Z)))
  (sum Z Z Z)
  (sum (S Z) Z (S Z))
  (sum (S (S Z)) (S Z) (S (S (S Z)))))
```

- *from-metafunction*: The first form randomly picks a term that satisfies the given invocation of the metafunction, using techniques similar to how the *from-judgment-form* case works. The second form uses a more naive approach; it simply generates terms that match the patterns of the cases of the metafunction; it does not consider the results of the metafunctions, nor does it consider patterns from earlier cases when generating terms based on a particular case. The third case is like the second, except it returns a function that accepts the size and keywords arguments that may be more efficient if multiple random terms are generated.

  Examples:

```
> (define-language L
    (n number))
> (define-metafunction L
    [(F one-clause n) ()]
    [(F another-clause n) ()])
> (for/list ([i (in-range 10)])
    (generate-term #:source F 5))
'((one-clause 3)
  (another-clause 2)
  (one-clause 1)
  (one-clause 3)
  (one-clause 0)
  (one-clause 1)
  (one-clause 4)
  (another-clause 2)
  (another-clause 0)
  (another-clause 1))
```

- *from-reduction-relation*: In the first case, `generate-term` randomly picks a rule from the reduction relation and tries to pick a term that satisfies its domain pattern, returning that. The second case returns a function that accepts the size and keyword arguments that may be more efficient if multiple random terms are generated.

  Examples:

```
> (define-language L
```

```
   (n number))
> (for/list ([i (in-range 10)])
    (generate-term
     #:source
     (reduction-relation
      L
      (--> (one-clause n) ())
      (--> (another-clause n) ()))
     5))
'((another-clause 0)
  (one-clause 1)
  (one-clause 0)
  (one-clause 2)
  (another-clause 0)
  (one-clause 0)
  (one-clause 1)
  (one-clause 0)
  (one-clause 1)
  (one-clause 0))
```

The argument *size-expr* bounds the height of the generated term (measured as the height of its parse tree).

The optional keyword argument *attempt-num-expr* (default 1) provides coarse grained control over the random decisions made during generation; increasing *attempt-num-expr* tends to increase the complexity of the result. For example, the absolute values of numbers chosen for integer patterns increase with *attempt-num-expr*.

The random generation process does not actively consider the constraints imposed by side-condition or _!_ patterns; instead, it uses a "guess and check" strategy in which it freely generates candidate terms then tests whether they happen to satisfy the constraints, repeating as necessary. The optional keyword argument *retries-expr* (default 100) bounds the number of times that generate-term retries the generation of any pattern. If generate-term is unable to produce a satisfying term after *retries-expr* attempts, it raises an exception recognized by exn:fail:redex:generation-failure?.

(redex-enum *language pattern*)

Constructs an enumeration that produces terms that match the given pattern, or #f if it cannot build an enumeration (which happens if the given pattern contains side-conditions).

It constructs a two-way enumeration only in some cases. The pattern must be unambiguous and there are other technical shortcomings of the implementation as well that cause the result to be a one-way enumeration in some situations.

Examples:

```
> (define-language L
    (e ::= (e e) x (λ (x) e))
    (x ::= variable-not-otherwise-mentioned))
> (from-nat (redex-enum L e) 38866548399079637577232342764876859 40)
'(λ (f) (f (f (f x))))
> (from-nat (redex-enum L e) 38866548399079637577232342764876859 42)
'(c (f (f (f x))))
> (from-nat (redex-enum L e) 38866548399079637577232342764876859 45)
'(((a a) a) (f (f (f x))))
> (to-nat (redex-enum L e)
          (term (λ (f) ((λ (x) (f (x x)))
                        (λ (x) (f (x x)))))))
6806924852705496960774096741475841654253439294593338486753906226879877500 5
```

(redex-index *language pattern term*)

Computes the index for an occurrence of the given term in the enumerator corresponding to the given pattern or returns `#f` if there is no enumerator.

This is useful when the pattern is ambiguous as you might still learn of an index that corresponds to the term even though the enumeration that `redex-enum` produces is a one-way enumeration.

Examples:

```
> (define-language L
    (e ::= (e e) x (λ (x) e))
    (x ::= variable-not-otherwise-mentioned))
> (redex-index L e
               (term (λ (f) ((λ (x) (f (x x)))
                             (λ (x) (f (x x)))))))
6806924852705496960774096741475841654253439294593338486753906226879877500 5
> (define-language L
    ; e is an ambiguous non-terminal
    ; because there are multiple ways to
    ; parse (cons (λ (x) x) (λ (x) x))
    (e ::= (e e) x (cons e e) v)
    (v ::= (cons v v) (λ (x) e))
    (x ::= variable-not-otherwise-mentioned))
> (redex-index L e
               (term ((λ (x) (x x))
                      (λ (x) (x x)))))
366600362279251777597
```

(redex-check *template property-expr kw-arg ...*)

```
template = language pattern
         | language pattern #:ad-hoc
         | language pattern #:in-order
         | language pattern #:uniform-at-random p-value
         | language pattern #:enum bound
         | language #:satisfying
           (judgment-form-id pattern ...)
         | language #:satisfying
           (metafunction-id pattern ...) = pattern

 kw-arg = #:attempts attempts-expr
        | #:source metafunction
        | #:source relation-expr
        | #:retries retries-expr
        | #:print? print?-expr
        | #:attempt-size attempt-size-expr
        | #:prepare prepare-expr
        | #:keep-going? keep-going?-expr

 property-expr : any/c
 attempts-expr : natural-number/c
 relation-expr : reduction-relation?
 retries-expr : natural-number/c
 print?-expr : any/c
 attempt-size-expr : (-> natural-number/c natural-number/c)
 prepare-expr : (-> any/c any/c)
```

Searches for a counterexample to *property-expr*, interpreted as a predicate universally quantified over the pattern variables bound by the pattern(s) in *template*. redex-check constructs and tests a candidate counterexample by choosing a random term *t* based on *template* and then evaluating *property-expr* using the match-bindings produced by matching *t* against pattern. The form of *template* controls how *t* is generated:

- *language pattern*: In this case, redex-check uses an ad hoc strategy for generating *pattern*. For the first half of the attempts (or only 500 if there are more than 1000 attempts) it uses in-order enumeration to pick terms. After that, it switches over to using the ad hoc random generator.
- *language pattern* #:ad-hoc: In this case, redex-check uses an ad hoc random generator to generate terms that match *pattern*.
- *language pattern* #:in-order: In this case, redex-check uses an enumeration

of *pattern*, checking each *t* one at a time

- *language pattern* `#:uniform-at-random` *p-value*: that to index into an enumeration of *pattern*. If the enumeration is finite, `redex-check` picks a natural number uniformly at random; if it isn't, `redex-check` uses a geometric distribution with *p-value* as its probability of zero to pick the number of bits in the index and then picks a number uniformly at random with that number of bits.

- *language pattern* `#:enum` *bound*: This is similar to `#:uniform-at-random`, except that Redex always picks a random natural number less than *bound* to index into the enumeration

- *language* `#:satisfying (`*judgment-form-id pattern* `...)`: Generates terms that match `pattern` and satisfy the judgment form.

- *language* `#:satisfying (`*metafunction-id pattern* `...) =` *pattern*: Generates terms matching the two `pattern`s, such that if the first is the argument to the metafunction, the second will be the result.

`redex-check` generates at most *attempts-expr* (default `(default-check-attempts)`) random terms in its search. The size and complexity of these terms tend to increase with each failed attempt. The `#:attempt-size` keyword determines the rate at which terms grow by supplying a function that bounds term size based on the number of failed attempts (see `generate-term`'s `size-expr` argument). By default, the bound grows according to the `default-attempt-size` function.

When *print?-expr* produces any non-`#f` value (the default), `redex-check` prints the test outcome on `current-output-port`. When *print?-expr* produces `#f`, `redex-check` prints nothing, instead

- returning a `counterexample` structure when the test reveals a counterexample,
- returning `#t` when all tests pass, or
- raising a `exn:fail:redex:test` when checking the property raises an exception.

The optional `#:prepare` keyword supplies a function that transforms each generated example before `redex-check` checks *property-expr*. This keyword may be useful when *property-expr* takes the form of a conditional, and a term chosen freely from the grammar is unlikely to satisfy the conditional's hypothesis. In some such cases, the `prepare` keyword can be used to increase the probability that an example satisfies the hypothesis.

The `#:retries` keyword behaves identically as in `generate-term`, controlling the number of times the generation of any pattern will be reattempted. It can't be used together with `#:satisfying`.

If *keep-going?-expr* produces any non-#f value, redex-check will stop only when it hits the limit on the number of attempts showing all of the errors it finds. This argument is allowed only when *print?-expr* is not #f.

When passed a metafunction or reduction relation via the optional #:source argument, redex-check distributes its attempts across the left-hand sides of that metafunction/relation by using those patterns, rather than pattern, as the basis of its generation. If any left-hand side generates a term that does not match pattern, then the test input is discarded. #:source cannot be used with #:satisfying. See also check-reduction-relation and check-metafunction.

Examples:

```
> (define-language empty-lang)
> (random-seed 0)
> (redex-check
   empty-lang
   ((number_1 ...)
    (number_2 ...))
   (equal? (reverse (append (term (number_1 ...))
                            (term (number_2 ...))))
           (append (reverse (term (number_1 ...)))
                   (reverse (term (number_2 ...))))))
redex-check: counterexample found after 11 attempts:
((+inf.0) (0))
> (redex-check
   empty-lang
   ((number_1 ...)
    (number_2 ...))
   (equal? (reverse (append (term (number_1 ...))
                            (term (number_2 ...))))
           (append (reverse (term (number_2 ...)))
                   (reverse (term (number_1 ...)))))
   #:attempts 200)
redex-check: no counterexamples in 200 attempts
> (let ([R (reduction-relation
            empty-lang
            (--> (Σ) 0)
            (--> (Σ number) number)
            (--> (Σ number_1 number_2 number_3 ...)
                 (Σ ,(+ (term number_1) (term number_2))
                    number_3 ...)))])
    (redex-check
     empty-lang
     (Σ number ...)
     (printf "~s\n" (term (number ...)))
```

```
     #:attempts 3
     #:source R))
()
(4)
(3 1)
redex-check: no counterexamples in 3 attempts (tried 1 attempt
with each clause)
> (redex-check
   empty-lang
   number
   (begin
     (printf "checking ~s\n" (term number))
     (positive? (term number)))
   #:prepare (λ (n)
               (printf "preparing ~s; " n)
               (add1 (abs (real-part n))))
   #:attempts 3)
preparing +inf.0; checking +inf.0
preparing 0; checking 1
preparing 0; checking 1
redex-check: no counterexamples in 3 attempts
> (define-language L
    (nat ::= Z (S nat)))
> (define-judgment-form L
    #:mode (sum I I O)
    [---------------
     (sum Z nat nat)]
    [(sum nat_1 nat_2 nat_3)
     ------------------------------
     (sum (S nat_1) nat_2 (S nat_3))])
> (redex-check L
               #:satisfying
               (sum nat_1 nat_2 nat_3)
               (equal? (judgment-holds
                        (sum nat_1 nat_2 nat_4) nat_4)
                       (term (nat_3)))
               #:attempts 100)
redex-check: no counterexamples in 100 attempts
> (redex-check L
               #:satisfying
               (sum nat_1 nat_2 nat_3)
               (equal? (term nat_1) (term nat_2)))
redex-check: counterexample found after 1 attempt:
(sum (S (S Z)) Z (S (S Z)))
```

Changed in version 1.10 of package `redex-lib`: Instead of raising an error, `redex-check` now discards test cases that don't match the given pattern when using `#:source`.

```
(depth-dependent-order?) → (or/c boolean? 'random)
(depth-dependent-order? depth-dependent) → void?
  depth-dependent : (or/c boolean? 'random)
 = 'random
```

Toggles whether or not Redex will dynamically adjust the chance that more recursive clauses of judgment forms or metafunctions are chosen earlier when attempting to generate terms with forms that use `#:satisfying`. If it is `#t`, Redex favors more recursive clauses at lower depths and less recursive clauses at depths closer to the limit, in an attempt to generate larger terms. When it is `#f`, all clause orderings have equal probability above the bound. By default, it is `'random`, which causes Redex to choose between the two above alternatives with equal probability.

```
(redex-generator language-id satisfying size-expr)

satisfying = (judgment-form-id pattern ...)
           | (metafunction-id pattern ...) = pattern

  size-expr : natural-number/c
```

*WARNING: `redex-generator` is a new, experimental form, and its API may change.*

Returns a thunk that, each time it is called, either generates a random s-expression based on *satisfying* or fails to (and returns `#f`). The terms returned by a particular thunk are guaranteed to be distinct.

Examples:

```
> (define-language L
    (nat ::= Z (S nat)))
> (define-judgment-form L
    #:mode (sum I I O)
    [---------------
     (sum Z nat nat)]
    [(sum nat_1 nat_2 nat_3)
     -------------------------------
     (sum (S nat_1) nat_2 (S nat_3))])
> (define gen-sum (redex-generator L (sum nat_1 nat_2 nat_3) 3))
> (for/list ([_ (in-range 5)])
    (gen-sum))
'((sum (S (S Z)) Z (S (S Z)))
  (sum (S (S (S (S Z)))) (S (S Z)) (S (S (S (S (S (S Z)))))))
  (sum (S (S (S (S (S Z))))) (S Z) (S (S (S (S (S (S Z)))))))
```

```
  (sum (S (S (S (S (S (S Z)))))) (S Z) (S (S (S (S (S (S (S
Z))))))))
  (sum (S (S (S (S (S (S (S Z))))))) Z (S (S (S (S (S (S (S
Z))))))))
```

```
(struct counterexample (term)
    #:extra-constructor-name make-counterexample
    #:transparent)
  term : any/c
```

Produced by `redex-check`, `check-reduction-relation`, and `check-metafunction` when testing falsifies a property.

```
(struct exn:fail:redex:test exn:fail:redex (source term)
    #:extra-constructor-name make-exn:fail:redex:test)
  source : exn:fail?
  term : any/c
```

Raised by `redex-check`, `check-reduction-relation`, and `check-metafunction` when testing a property raises an exception. The `exn:fail:redex:test-source` component contains the exception raised by the property, and the `exn:fail:redex:test-term` component contains the term that induced the exception.

```
(check-reduction-relation relation property kw-args ...)

kw-arg = #:attempts attempts-expr
       | #:retries retries-expr
       | #:print? print?-expr
       | #:attempt-size attempt-size-expr
       | #:prepare prepare-expr

  property : (-> any/c any/c)
  attempts-expr : natural-number/c
  retries-expr : natural-number/c
  print?-expr : any/c
  attempt-size-expr : (-> natural-number/c natural-number/c)
  prepare-expr : (-> any/c any/c)
```

Tests *relation* as follows: for each case of *relation*, `check-reduction-relation` generates `attempts` random terms that match that case's left-hand side and applies *property* to each random term.

Only the primary pattern of each case's left-hand side is considered. If there are `where` clauses or `side-conditions` (or anything else from the *red-extras* portion of the grammar), they are ignored.

This form provides a more convenient notation for

```
(redex-check L any (property (term any))
             #:attempts (* n attempts)
             #:source relation)
```

when *relation* is a relation on L with n rules.

```
(check-metafunction metafunction property kw-args ...)

kw-arg = #:attempts attempts-expr
       | #:retries retries-expr
       | #:print? print?-expr
       | #:attempt-size attempt-size-expr
       | #:prepare prepare-expr

  property : (-> (listof any/c) any/c)
  attempts-expr : natural-number/c
  retries-expr : natural-number/c
  print?-expr : any/c
  attempt-size-expr : (-> natural-number/c natural-number/c)
  prepare-expr : (-> (listof any/c) (listof any/c))
```

Like `check-reduction-relation` but for metafunctions. `check-metafunction` calls *property* with lists containing arguments to the metafunction. Similarly, *prepare-expr* produces and consumes argument lists.

Only the primary pattern of each case's left-hand side is considered. If there are `where` clauses or `side-conditions` (or anything else from the *metafunction-extras* portion of the grammar), they are ignored.

Examples:

```
> (define-language empty-lang)
> (define-metafunction empty-lang
    Σ : number ... -> number
    [(Σ) 0]
    [(Σ number) number]
    [(Σ number_1 number_2) ,(+ (term number_1) (term number_2))]
    [(Σ number_1 number_2 ...) (Σ number_1 (Σ number_2 ...))])
> (check-metafunction Σ (λ (args)
                          (printf "trying ~s\n" args)
                          (equal? (apply + args)
                                  (term (Σ ,@args))))
                      #:attempts 2)
```

```
trying ()
trying ()
trying (0)
trying (0)
trying (2 1)
trying (0 1)
trying (0)
trying (1)
check-metafunction: no counterexamples in 8 attempts (tried 2 at-
tempts with each clause)
```

```
(default-attempt-size n) → natural-number/c
  n : natural-number/c
```

The default value of the `#:attempt-size` argument to `redex-check` and the other randomized testing forms, this procedure computes an upper bound on the size of the next test case from the number of previously attempted tests *n*. Currently, this procedure computes the base 5 logarithm, but that behavior may change in future versions.

```
(default-check-attempts) → natural-number/c
(default-check-attempts attempts) → void?
  attempts : natural-number/c
```

Determines the default value for `redex-check`'s optional `#:attempts` argument. By default, *attempts* is 1,000.

```
(redex-pseudo-random-generator) → pseudo-random-generator?
(redex-pseudo-random-generator generator) → void?
  generator : pseudo-random-generator?
```

`generate-term` and the randomized testing forms (e.g., `redex-check`) use the parameter *generator* to construct random terms. The parameter's initial value is `(current-pseudo-random-generator)`.

```
(exn:fail:redex:generation-failure? v) → boolean?
  v : any/c
```

Recognizes the exceptions raised by `generate-term`, `redex-check`, etc. when those forms are unable to produce a term matching some pattern.

*Debugging PLT Redex Programs*

It is easy to write grammars and reduction rules that are subtly wrong. Typically such mistakes result in examples that get stuck when viewed in a `traces` window.

The best way to debug such programs is to find an expression that looks like it should reduce, but doesn't, then try to find out which pattern is failing to match. To do so, use the `redex-match` form.

In particular, first check if the term in question matches the your system's main non-terminal (typically the expression or program non-terminal). If it does not match, simplify the term piece by piece to determine whether the problem is in the term or the grammar.

If the term does match your system's main non-terminal, determine by inspection which reduction rules should apply. For each such rule, repeat the above term-pattern debugging procedure, this time using the rule's left-hand side pattern instead of the system's main non-terminal. In addition to simplifying the term, also consider simplifying the pattern.

If the term matches the left-hand side, but the rule does not apply, then one of the rule's `side-condition` or `where` clauses is not satisfied. Using the bindings reported by `redex-match`, check each `side-condition` expression and each `where` pattern-match to discover which clause is preventing the rule's application.

## 4.7 GUI

`(require redex/gui)` package: `redex-gui-lib`

This section describes the GUI tools that Redex provides for exploring reduction sequences.

```
(traces  reductions
         expr
        [#:multiple? multiple?
         #:reduce reduce
         #:pred pred
         #:pp pp
         #:colors colors
         #:racket-colors? racket-colors?
         #:scheme-colors? scheme-colors?
         #:filter term-filter
         #:x-spacing x-spacing
         #:y-spacing y-spacing
         #:layout layout
         #:edge-labels? edge-labels?
         #:edge-label-font edge-label-font
         #:autorun-dot? autorun-dot?
         #:hide-controls? hide-controls?
         #:graph-pasteboard-mixin graph-pasteboard-mixin])
 → void?
  reductions : (or/c reduction-relation? IO-judgment-form?)
  expr : (or/c any/c (listof any/c))
  multiple? : boolean? = #f
```

```
reduce : (-> reduction-relation? any/c
             (listof (list/c (union false/c string?) any/c)))
       = apply-reduction-relation/tag-with-names
pred : (or/c (-> sexp any)              = (λ (x) #t)
             (-> sexp term-node? any))
pp : (or/c (any -> string)
           (any output-port number (is-a?/c text%) -> void))
   = default-pretty-printer
colors : (listof
          (cons/c string?
                  (and/c (listof (or/c string? (is-a?/c color%)))
                         (λ (x) (<= 0 (length x) 6)))))
       = '()
racket-colors? : boolean? = #t
scheme-colors? : boolean? = racket-colors?
term-filter : (-> any/c (or/c #f string?) any/c)
            = (λ (x y) #t)
x-spacing : real? = 15
y-spacing : real? = 15
layout : (-> (listof term-node?) void?) = void
edge-labels? : boolean? = #t
edge-label-font : (or/c #f (is-a?/c font%)) = #f
autorun-dot? : any/c = #f
hide-controls? : any/c = #f
graph-pasteboard-mixin : (make-mixin-contract graph-pasteboard<%>)
                       = values
```

This function opens a new window and inserts each expression in *expr* (if *multiple?* is #t – if *multiple?* is #f, then *expr* is treated as a single expression). Then, it reduces the terms until at least reduction-steps-cutoff (see below) different terms are found, or no more reductions can occur. It inserts each new term into the gui. Clicking the reduce button reduces until reduction-steps-cutoff more terms are found.

The *reduce* function applies the reduction relation to the terms. By default, it is apply-reduction-relation/tag-with-names; it may be changed to only return a subset of the possible reductions, for example, but it must satisfy the same contract as apply-reduction-relation/tag-with-names.

If *reductions* is an IO-judgment-form?, then the judgment form is treated as a reduction relation. The initial input position is the given *expr* and the output position becomes the next input.

The *pred* function indicates if a term has a particular property. If it returns #f, the term is displayed with a pink background. If it returns a string or a color% object, the term is displayed with a background of that color (using the-color-database to map the string to a color). If it returns any other value, the term is displayed normally. If the *pred* function

accepts two arguments, a term-node corresponding to the term is passed to the predicate. This lets the predicate function explore the (names of the) reductions that led to this term, using `term-node-children`, `term-node-parents`, and `term-node-labels`.

The *pred* function may be called more than once per node. In particular, it is called each time an edge is added to a node. The latest value returned determines the color.

The *pp* function is used to specially print expressions. It must either accept one or four arguments. If it accepts one argument, it will be passed each term and is expected to return a string to display the term.

If the *pp* function takes four arguments, it should render its first argument into the port (its second argument) with width at most given by the number (its third argument). The final argument is the text where the port is connected – characters written to the port go to the end of the editor. Use `write-special` to send `snip%` objects or `2htdp/image` images (or other things that subscribe to `file/convertible` or `pict/convert`) directly to the editor.

The *colors* argument, if provided, specifies a list of reduction-name/color-list pairs. The traces gui will color arrows drawn because of the given reduction name with the given color instead of using the default color.

The `cdr` of each of the elements of *colors* is a list of colors, organized in pairs. The first two colors cover the colors of the line and the border around the arrow head, the first when the mouse is over a graph node that is connected to that arrow, and the second for when the mouse is not over that arrow. Similarly, the next colors are for the text drawn on the arrow and the last two are for the color that fills the arrow head. If fewer than six colors are specified, the specified colors are used and then defaults are filled in for the remaining colors.

The *racket-colors?* argument (along with *scheme-colors?*, retained for backward compatibility), controls the coloring of each window. When *racket-colors?* is `#t` (and *scheme-colors?* is `#t` too), `traces` colors the contents according to DrRacket's Racket-mode color scheme; otherwise, `traces` uses a black color scheme.

The *term-filter* function is called each time a new node is about to be inserted into the graph. If the filter returns false, the node is not inserted into the graph.

The *x-spacing* and *y-spacing* arguments control the amount of space put between the snips in the default layout.

The *layout* argument is called (with all of the terms) when new terms are inserted into the window. In general, it is called after new terms are inserted in response to the user clicking on the reduce button, and after the initial set of terms is inserted. See also `term-node-set-position!`.

If *edge-labels?* is `#t` (the default), then edge labels are drawn; otherwise not.

The *edge-label-font* argument is used as the font on the edge labels. If #f is supplied, the dc<%> object's default font is used.

If *autorun-dot?* is a true value and the dot binary can be found, then the layout is adjusted using dot before the window is shown.

If *hide-controls?* is a true value, then the controls on the bottom of the window are initially hidden.

The traces library uses an instance of the mrlib/graph library's graph-pasteboard<%> interface to layout the graphs. Sometimes, overriding one of its methods can help give finer-grained control over the layout, so the *graph-pasteboard-mixin* is applied to the class before it is instantiated. Also note that all of the snips inserted into the editor by this library have a get-term-node method which returns the snip's term-node?.

The Fix Layout button calls out to dot (a program that's a part of graphviz) and uses its results to lay out the graph. While in the fixed-layout mode, the edges are now drawn. Click the button again to restore the edges.

For a more serious example of traces, please see §1 "Amb: A Redex Tutorial", but for a silly one that demonstrates how the *pp* argument lets us use images, we can take the pairing functions discussed in Matthew Szudzik's *An Elegant Pairing Function* presentation:

```
(define/contract (unpair z)
  (-> exact-nonnegative-integer?
      (list/c exact-nonnegative-integer? exact-nonnegative-
integer?))
  (define i (integer-sqrt z))
  (define i2 (* i i))
  (cond
    [(< (- z i2) i)
     (list (- z i2) i)]
    [else
     (list i (- z i2 i))]))

(define/contract (pair x y)
  (-> exact-nonnegative-integer? exact-nonnegative-integer?
      exact-nonnegative-integer?)
  (if (= x (max x y))
      (+ (* x x) x y)
      (+ (* y y) x)))
```

and build a reduction relation out of them:

```
(define-language L (n ::= natural))
(define red
  (reduction-relation
```

```
   L
   (--> (n_1 n_2)
        ,(unpair (+ 1 (pair (term n_1)
                            (term n_2)))))))
(traces red (term (0 0)))
```

We can then turn those two numbers into two stars, where the number indicates the number of points in the star:

```
(require 2htdp/image)
(define/contract (two-stars point-count1 point-count2)
  (-> (>=/c 2) (>=/c 2) image?)
  (overlay
   (radial-star (+ 2 point-count1)
                10 60
                "solid"
                (make-color 255 0 255 150))
   (radial-star (+ 2 point-count2)
                10 60
                "solid"
                "cornflowerblue")))
```

and then use the *pp* function to show those in the traces window instead of just the numbers.

```
(traces red
        (term (0 0))
        #:pp
        (λ (term port w txt)
          (write-special
           (two-stars (+ 2 (list-ref term 0))
                      (+ 2 (list-ref term 1)))
           port)))
```

```
(traces/ps  reductions
            expr
            file
           [#:multiple? multiple?
            #:reduce reduce
            #:pred pred
            #:pp pp
            #:colors colors
            #:filter term-filter
            #:layout layout
            #:x-spacing x-spacing
            #:y-spacing y-spacing
            #:edge-labels? edge-labels?
            #:edge-label-font edge-label-font
            #:graph-pasteboard-mixin graph-pasteboard-mixin]
            #:post-process post-process)
 → void?
  reductions : (or/c reduction-relation? IO-judgment-form?)
  expr : (or/c any/c (listof any/c))
  file : (or/c path-string? path?)
  multiple? : boolean? = #f
  reduce : (-> reduction-relation? any/c
               (listof (list/c (union false/c string?) any/c)))
         = apply-reduction-relation/tag-with-names
  pred : (or/c (-> sexp any)              = (λ (x) #t)
               (-> sexp term-node? any))
  pp : (or/c (any -> string)
             (any output-port number (is-a?/c text%) -> void))
     = default-pretty-printer
  colors : (listof
            (cons/c string?
                    (and/c (listof (or/c string? (is-a?/c color%)))
                           (λ (x) (<= 0 (length x) 6)))))
         = '()
  term-filter : (-> any/c (or/c #f string?) any/c)
              = (λ (x y) #t)
  layout : (-> (listof term-node?) void?) = void
  x-spacing : number? = 15
  y-spacing : number? = 15
  edge-labels? : boolean? = #t
  edge-label-font : (or/c #f (is-a?/c font%)) = #f
  graph-pasteboard-mixin : (make-mixin-contract graph-pasteboard<%>)
                         = values
  post-process : (-> (is-a?/c graph-pasteboard<%>) any/c)
```

This function behaves just like the function traces, but instead of opening a window to show the reduction graph, it just saves the reduction graph to the specified *file*.

All of the arguments behave like the arguments to traces, with the exception of the *post-process* argument. It is called just before the PostScript is created with the graph pasteboard.

```
(stepper  reductions
          t
         [pp
          #:show-font-size-control? show-font-size-control?])
 → void?
  reductions : (or/c reduction-relation? IO-judgment-form?)
  t : any/c
  pp : (or/c (any -> string)
             (any output-port number (is-a?/c text%) -> void))
     = default-pretty-printer
  show-font-size-control? : any/c = #f
```

This function opens a stepper window for exploring the behavior of the term *t* in the reduction system given by *reductions*.

The *pp* argument is the same as to the traces function but is here for backwards compatibility only and should not be changed for most uses, but instead adjusted with pretty-print-parameters. Specifically, the highlighting shown in the stepper window can be wrong if default-pretty-printer does not print sufficiently similarly to how pretty-print prints (when adjusted by pretty-print-parameters's behavior, of course).

If *show-font-size-control?* is a true value, then a slider with a font size choice is shown in the window.

Changed in version 1.1 of package redex-gui-lib: Added the *show-font-size-control?* argument.

```
(stepper/seed
  reductions
  seed
 [pp
  #:show-font-size-control? show-font-size-control?])
→ void?
 reductions : (or/c reduction-relation? IO-judgment-form?)
 seed : (cons/c any/c (listof any/c))
 pp : (or/c (any -> string)
            (any output-port number (is-a?/c text%) -> void))
    = default-pretty-printer
 show-font-size-control? : any/c = #f
```

Like stepper, this function opens a stepper window, but it seeds it with the reduction-

sequence supplied in *seed*.

Changed in version 1.1 of package `redex-gui-lib`: Added the *show-font-size-control?* argument.

```
(show-derivations  derivations
                   [#:pp pp
                    #:racket-colors? racket-colors?
                    #:init-derivation init-derivation]) → any
  derivations : (cons/c derivation? (listof derivation?))
  pp : (or/c (any -> string)
             (any output-port number (is-a?/c text%) -> void))
     = default-pretty-printer
  racket-colors? : boolean? = #f
  init-derivation : exact-nonnegative-integer? = 0
```

Opens a window to show *derivations*.

The *pp* and *racket-colors?* arguments are like those to `traces`.

The initial derivation shown in the window is chosen by *init-derivation*, used as an index into *derivations*.

```
(derivation/ps  derivation
                filename
                [#:pp pp
                 #:racket-colors? racket-colors?]
                 #:post-process post-process)     → void?
  derivation : derivation?
  filename : path-string?
  pp : (or/c (any -> string)
             (any output-port number (is-a?/c text%) -> void))
     = default-pretty-printer
  racket-colors? : boolean? = #f
  post-process : (-> (is-a?/c pasteboard%) any)
```

Like `show-derivations`, except it prints a single derivation in PostScript to *filename*.

```
(term-node-children tn) → (listof term-node?)
  tn : term-node?
```

Returns a list of the children (i.e., terms that this term reduces to) of the given node.

Note that this function does not return all terms that this term reduces to – only those that are currently in the graph.

```
(term-node-parents tn) → (listof term-node?)
  tn : term-node?
```

Returns a list of the parents (i.e., terms that reduced to the current term) of the given node.

Note that this function does not return all terms that reduce to this one – only those that are currently in the graph.

```
(term-node-labels tn) → (listof (or/c false/c string?))
  tn : term-node?
```

Returns a list of the names of the reductions that led to the given node, in the same order as the result of term-node-parents. If the list contains `#f`, that means that the corresponding step does not have a label.

```
(term-node-set-color! tn color) → void?
  tn : term-node?
  color : (or/c string? (is-a?/c color%) false/c)
```

Changes the highlighting of the node; if its second argument is `#f`, the coloring is removed, otherwise the color is set to the specified `color%` object or the color named by the string. The `color-database<%>` is used to convert the string to a `color%` object.

```
(term-node-color tn) → (or/c string? (is-a?/c color%) false/c)
  tn : term-node?
```

Returns the current highlighting of the node. See also `term-node-set-color!`.

```
(term-node-set-red! tn red?) → void?
  tn : term-node?
  red? : boolean?
```

Changes the highlighting of the node; if its second argument is `#t`, the term is colored pink, if it is `#f`, the term is not colored specially.

```
(term-node-expr tn) → any
  tn : term-node?
```

Returns the expression in this node.

```
(term-node-set-position! tn x y) → void?
  tn : term-node?
  x : (and/c real? positive?)
  y : (and/c real? positive?)
```

Sets the position of `tn` in the graph to (`x`,`y`).

```
(term-node-x tn) → real?
  tn : term-node?
```

Returns the x coordinate of *tn* in the window.

```
(term-node-y tn) → real?
  tn : term-node?
```

Returns the y coordinate of *tn* in the window.

```
(term-node-width tn) → real?
  tn : term-node?
```

Returns the width of *tn* in the window.

```
(term-node-height tn) → real?
  tn : term-node?
```

Returns the height of *tn* in the window.

```
(term-node? v) → boolean?
  v : any/c
```

Recognizes term nodes.

```
(reduction-steps-cutoff) → natural-number/c
(reduction-steps-cutoff cutoff) → void?
  cutoff : natural-number/c
```

A parameter that controls how many steps the `traces` function takes before stopping.

```
(initial-font-size) → number?
(initial-font-size size) → void?
  size : number?
```

A parameter that controls the initial font size for the terms shown in the GUI window.

```
(initial-char-width) → (or/c number? (-> any/c number?))
(initial-char-width width) → void?
  width : (or/c number? (-> any/c number?))
```

A parameter that determines the initial width of the boxes where terms are displayed (measured in characters) for both the stepper and traces.

If its value is a number, then the number is used as the width for every term. If its value is a function, then the function is called with each term and the resulting number is used as the width.

```
(dark-pen-color) → (or/c string? (is-a?/c color<%>))
(dark-pen-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
(dark-brush-color) → (or/c string? (is-a?/c color<%>))
(dark-brush-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
(light-pen-color) → (or/c string? (is-a?/c color<%>))
(light-pen-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
(light-brush-color) → (or/c string? (is-a?/c color<%>))
(light-brush-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
(dark-text-color) → (or/c string? (is-a?/c color<%>))
(dark-text-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
(light-text-color) → (or/c string? (is-a?/c color<%>))
(light-text-color color) → void?
  color : (or/c string? (is-a?/c color<%>))
```

These six parameters control the color of the edges in the graph.

The dark colors are used when the mouse is over one of the nodes that is connected to this edge. The light colors are used when it isn't.

The pen colors control the color of the line. The brush colors control the color used to fill the arrowhead and the text colors control the color used to draw the label on the edge.

```
(pretty-print-parameters) → (-> (-> any/c) any/c)
(pretty-print-parameters f) → void?
  f : (-> (-> any/c) any/c)
```

A parameter that is used to set other `pretty-print` parameters.

Specifically, whenever `default-pretty-printer` prints something it calls `f` with a thunk that does the actual printing. Thus, `f` can adjust `pretty-print`'s parameters to adjust how printing happens.

```
(default-pretty-printer v port width text) → void?
  v : any/c
  port : output-port?
```

```
width : exact-nonnegative-integer?
text : (is-a?/c text%)
```

This is the default value of pp used by traces and stepper and it uses pretty-print.

This function uses the value of pretty-print-parameters to adjust how it prints.

It sets the pretty-print-columns parameter to *width*, and it sets pretty-print-size-hook and pretty-print-print-hook to print holes and the symbol 'hole to match the way they are input in a term expression.

## 4.8 Typesetting

(require redex/pict) package: redex-pict-lib

The redex/pict library provides functions designed to typeset grammars, reduction relations, and metafunctions.

Each grammar, reduction relation, and metafunction, and even individual examples can be turned into picts for use with Scribble or Slideshow.

For producing papers with Scribble, just include the picts inline in the paper. For producing papers with LaTeX, create ".ps" files from the picts with Redex, which can be included into a LaTeX document.

When Rhombus pict has been loaded (specifically, when (module-declared? '(lib "pict/main.rhm") #t) returns a true value), the redex/pict library creates (and accepts) Rhombus picts. See the Rhombus Pict documentation for more information about Rhombus picts.

Changed in version 1.10 of package redex-pict-lib: Added support for Rhombus picts.

### 4.8.1 Generating Picts

This section documents two sets of operations, one for direct use of creating figures for use in papers and for use in DrRacket to easily adjust the typesetting: render-term, render-language, render-reduction-relation, render-relation, render-judgment-form, render-metafunctions, and render-lw, and a second set for use in combination with other libraries that operate on picts term->pict, language->pict, reduction-relation->pict, relation->pict, judgment-form->pict, derivation->pict, metafunction->pict, and lw->pict. The primary difference between these functions is that the former sets dc-for-text-size and the latter does not.

```
(render-term lang term)
(render-term lang term file)
```

Renders the term *term*. If *file* is `#f` or not present, `render-term` produces a pict; if *file* is a path, it saves Encapsulated PostScript in the provided filename, unless the filename ends with `".pdf"`, in which case it saves PDF.

Examples:

```
> (define-language nums
    (AE K
        (+ AE AE))
    ; binary digits
    (k 1 0)
    ; binary constants
    (K · (k K)))
> (render-term nums (+ (1 (0 (1 ·))) (+ (1 (1 (1 ·))) (1 (0 (0 ·))))))
```
(+ (1 (0 (1 ·))) (+ (1 (1 (1 ·))) (1 (0 (0 ·)))))

The *term* argument must be a literal; it is not an evaluated position. For example:

Example:

```
> (let ([x (term (+ (1 (1 (1 ·))) (1 (0 (0 ·)))))])
    (render-term nums x))
```
x

but also see `render-term/pretty-write`.

See `render-language` for more details on the construction of the pict.

Changed in version 1.16 of package `redex-pict-lib`: Changed how `in-hole` renders when its second argument is `hole`, avoiding a special case for that situation.

```
(term->pict lang term)
```

Produces a pict like `render-term`, but without adjusting `dc-for-text-size`.

The first argument is expected to be a `compiled-lang?` and the second argument is expected to be a term (without the *term* wrapper). The formatting in the *term* argument is used to determine how the resulting pict will look.

This function is primarily designed to be used with Slideshow or with other tools that combine `pict`s together.

Example:

```
> (term->pict nums (+ 1 (+ 3 4)))
(+ 1 (+ 3 4))
```

```
(render-term/pretty-write lang
                          term
                          [filename
                           #:width width])
 → (or/c void? pict-convertible?)
  lang : compiled-lang?
  term : any/c
  filename : (or/c path-string? #f) = #f
  width : (or/c exact-positive-integer? 'infinity)
        = (pretty-print-columns)
```

Like `render-term`, except that the *term* argument is evaluated, and expected to return a term. Then, `pretty-write` is used to determine where the line breaks go, using the *width* argument as a maximum width (via `pretty-print-columns`).

If *filename* is provided, the pict is saved as a pdf to that file.

Example:

```
> (render-term/pretty-write nums '(+ (1 1 1) (1 0 1)))
(+ (1 1 1) (1 0 1))
```

```
(term->pict/pretty-write lang
                         term
                         [#:width width]) → pict-convertible?
  lang : compiled-lang?
  term : any/c
  width : (or/c exact-positive-integer? 'infinity)
        = (pretty-print-columns)
```

Like `term->pict`, but with the same change that `render-term/pretty-write` has from `render-term`.

Example:

```
> (term->pict/pretty-write nums '(+ (1 1 1) (1 0 1)))
(+ (1 1 1) (1 0 1))
```

```
(render-language lang [file #:nts nts])
 → (if file void? pict-convertible?)
  lang : compiled-lang?
  file : (or/c #f path-string?) = #f
  nts : (or/c #f (listof (or/c string? symbol?)))
      = (render-language-nts)
```

Renders a language. If *file* is #f, it produces a pict; if *file* is a path, it saves Encapsulated PostScript in the provided filename, unless the filename ends with ".pdf", in which case it saves PDF. See render-language-nts for information on the *nts* argument.

This function parameterizes dc-for-text-size to install a relevant dc: a bitmap-dc% or a post-script-dc%, depending on whether *file* is a path.

See language->pict if you are using Slideshow or are otherwise setting dc-for-text-size.

Example:

```
> (render-language nums)
AE ::= K
     | (+ AE AE)
  k ::= 1 | 0
  K ::= · | (k K)
```

```
(language->pict lang [#:nts nts]) → pict-convertible?
  lang : compiled-lang?
  nts : (or/c #f (listof (or/c string? symbol?)))
      = (render-language-nts)
```

Produce a pict like render-language, but without adjusting dc-for-text-size.

This function is primarily designed to be used with Slideshow or with other tools that combine picts together.

Example:

```
> (language->pict nums)
AE ::= K
     | (+ AE AE)
  k ::= 1 | 0
  K ::= · | (k K)
```

```
(render-reduction-relation  rel
                            [file
                             #:style style])
 → (if file void? pict-convertible?)
  rel : reduction-relation?
  file : (or/c #f path-string?) = #f
  style : reduction-rule-style/c = (rule-pict-style)
```

Renders a reduction relation. If *file* is #f, it produces a pict; if *file* is a path, it saves Encapsulated PostScript in the provided filename, unless the filename ends with ".pdf", in which case it saves PDF. See rule-pict-style for information on the *style* argument.

This function parameterizes dc-for-text-size to install a relevant dc: a bitmap-dc% or a post-script-dc%, depending on whether *file* is a path. See also reduction-relation->pict.

The following forms of arrows can be typeset:

```
-->  -+>  ==>  ->  =>  ..>  >->  ~~>  ~>  :->  :-->  c->  -->>
 >--  --<  >>--  --<<
```

Examples:

```
> (define simplify-ae
    (reduction-relation
     nums
     (--> (+ AE ())
          AE)
     (--> (+ AE_1
             AE_2)
          (+ AE_2
             AE_1))))
> (render-reduction-relation
   simplify-ae)
```

$(+\ AE\ ()) \longrightarrow AE$

$(+\ AE_1\ AE_2) \longrightarrow (+\ AE_2\ AE_1)$

```
(reduction-relation->pict r [#:style style]) → pict-convertible?
  r : reduction-relation?
  style : reduction-rule-style/c = (rule-pict-style)
```

Produces a pict like render-reduction-relation, but without setting dc-for-text-size.

This function is primarily designed to be used with Slideshow or with other tools that combine picts together.

Example:

```
> (reduction-relation->pict
    (reduction-relation
     nums
```

```
     (--> (+ (+ AE_1 AE_2) AE_3)
          (+ AE_1 (+ AE_2 AE_3)))))
```
$(+\ (+\ AE_1\ AE_2)\ AE_3) \longrightarrow$
$(+\ AE_1\ (+\ AE_2\ AE_3))$

```
(render-metafunction metafunction-name maybe-contract)
(render-metafunction metafunction-name filename maybe-contract)
(render-metafunctions metafunction-name ...
                      maybe-filename maybe-contract maybe-only-contract)

     maybe-filename =
                    | #:file filename
                    | #:filename filename

    maybe-contract? =
                    | #:contract? bool-expr

maybe-only-contract? =
                    | #:only-contract? bool-expr
```

Like `render-reduction-relation` but for metafunctions.

Similarly, `render-metafunctions` accepts multiple metafunctions and renders them together, lining up all of the clauses together.

There are a number of different styles that affect the overall rendering of the metafunction, controlled by `metafunction-pict-style`. Other parameters that affect rendering include `linebreaks`, `sc-linebreaks`, and `metafunction-cases`.

If the metafunctions have contracts, they are typeset as the first lines of the output unless the expression following `#:contract?` evaluates to `#f` (which is the default). If the expression following `#:only-contract?` is not `#false` (the default) then only the contract is typeset.

This function sets `dc-for-text-size`. See also `metafunction->pict` and `metafunctions->pict`.

Examples:

```
> (define-metafunction nums
    add : K K -> K
    [(add K ·) K]
    [(add · K) K]
    [(add (0 K_1) (0 K_2)) (0 (add K_1 K_2))]
    [(add (1 K_1) (0 K_2)) (1 (add K_1 K_2))]
    [(add (0 K_1) (1 K_2)) (1 (add K_1 K_2))]
    [(add (1 K_1) (1 K_2)) (0 (add (1 ·) (add K_1 K_2)))])
```

```
> (render-metafunction add #:contract? #t)
```
$$\begin{array}{l}\mathsf{add} : K\ K \to K \\ \mathsf{add}[\![K, \cdot]\!] = K \\ \mathsf{add}[\![\cdot, K]\!] = K \\ \mathsf{add}[\![(0\ K_1), (0\ K_2)]\!] = (0\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(1\ K_1), (0\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(0\ K_1), (1\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(1\ K_1), (1\ K_2)]\!] = (0\ (\mathsf{add}\ (1\ \cdot)\ (\mathsf{add}\ K_1\ K_2)))\end{array}$$

Changed in version 1.3 of package `redex-pict-lib`: Added `#:contract?` keyword argument.
Changed in version 1.7: Added `#:only-contract?` keyword argument.

```
(metafunction->pict metafunction-name maybe-contract? maybe-only-
contract?)
```

Produces a pict like `render-metafunction`, but without setting `dc-for-text-size`. It is suitable for use in Slideshow or other libraries that combine `pict`s.

Example:

```
> (metafunction->pict add)
```
$$\begin{array}{l}\mathsf{add}[\![K, \cdot]\!] = K \\ \mathsf{add}[\![\cdot, K]\!] = K \\ \mathsf{add}[\![(0\ K_1), (0\ K_2)]\!] = (0\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(1\ K_1), (0\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(0\ K_1), (1\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\ \mathsf{add}[\![(1\ K_1), (1\ K_2)]\!] = (0\ (\mathsf{add}\ (1\ \cdot)\ (\mathsf{add}\ K_1\ K_2)))\end{array}$$

Changed in version 1.3 of package `redex-pict-lib`: Added `#:contract?` keyword argument.
Changed in version 1.7: Added `#:only-contract?` keyword argument.

```
(metafunctions->pict metafunction-name ...)
```

Like `metafunction->pict`, this produces a `pict`, but without setting `dc-for-text-size` and is suitable for use in Slideshow or other libraries that combine `pict`s. Like `render-metafunctions`, it accepts multiple metafunctions and renders them together.

Example:

```
> (define-metafunction nums
    to-nat : K -> natural
    [(to-nat ·) 0]
    [(to-nat (0 K)) ,(* 2 (term (to-nat K)))]
    [(to-nat (1 K)) ,(+ 1 (* 2 (term (to-nat K))))])
```

Example:

```
> (metafunctions->pict add to-nat)
```
add⟦$K$, ·⟧ = $K$
add⟦·, $K$⟧ = $K$
add⟦(0 $K_1$), (0 $K_2$)⟧ = (0 (add $K_1$ $K_2$))
add⟦(1 $K_1$), (0 $K_2$)⟧ = (1 (add $K_1$ $K_2$))
add⟦(0 $K_1$), (1 $K_2$)⟧ = (1 (add $K_1$ $K_2$))
add⟦(1 $K_1$), (1 $K_2$)⟧ = (0 (add (1 ·) (add $K_1$ $K_2$)))
to-nat⟦·⟧ = 0
to-nat⟦(0 $K$)⟧ = `(* 2` (to-nat $K$)`)`
to-nat⟦(1 $K$)⟧ = `(+ 1 (* 2` (to-nat $K$)`))`

```
(render-relation relation-name)
(render-relation relation-name filename)
```

Like `render-metafunction` but for relations.

This function sets `dc-for-text-size`. See also `relation->pict`.

```
(render-judgment-form judgment-form
                      filename)     → pict?
  judgment-form : judgment-form?
  filename : (or/c path-string? #f)
```

Like `render-metafunction` but for judgment forms. The `judgment-form-cases` parameter can be used to control which clauses are rendered.

Examples:

```
> (define-judgment-form nums
    #:mode (eq I I)
    #:contract (eq K K)

    [--------- eq-··
     (eq · ·)]

    [(eq K ·)
     ------------ eq-0-l
     (eq (0 K) ·)]

    [(eq · K)
     ------------ eq-0-r
     (eq · (0 K))]
```

```
    [(eq K_1 K_2)
     -------------------- eq-0
     (eq (0 K_1) (0 K_2))]

    [(eq K_1 K_2)
     -------------------- eq-1
     (eq (1 K_1) (1 K_2))])
> (render-judgment-form eq)
```

$$\frac{}{\mathsf{eq}[\![\cdot, \cdot]\!]}\,[\text{eq-}\cdot\cdot]$$

$$\frac{\mathsf{eq}[\![K, \cdot]\!]}{\mathsf{eq}[\![(0\ K), \cdot]\!]}\,[\text{eq-0-l}]$$

$$\frac{\mathsf{eq}[\![\cdot, K]\!]}{\mathsf{eq}[\![\cdot, (0\ K)]\!]}\,[\text{eq-0-r}]$$

$$\frac{\mathsf{eq}[\![K_1, K_2]\!]}{\mathsf{eq}[\![(0\ K_1), (0\ K_2)]\!]}\,[\text{eq-0}]$$

$$\frac{\mathsf{eq}[\![K_1, K_2]\!]}{\mathsf{eq}[\![(1\ K_1), (1\ K_2)]\!]}\,[\text{eq-1}]$$

```
> (parameterize ([judgment-form-cases '("eq-·")])
    (render-judgment-form eq))
```

$$\frac{}{\mathsf{eq}[\![\cdot, \cdot]\!]}\,[\text{eq-}\cdot\cdot]$$

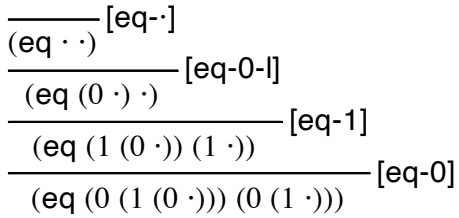
This function sets `dc-for-text-size`. See also `judgment-form->pict`.

```
(derivation->pict language derivation) → pict-convertible?
  language : compiled-lang?
  derivation : derivation?
```

Produces a pict that looks like the derivation in `show-derivations`, except that it uses `term->pict/pretty-write` to draw the individual terms in the derivation.

Example:

```
> (derivation->pict nums (car (build-derivations (eq (0 (1 (0 ·))) (0 (1 ·)))))))
```




Added in version 1.8 of package `redex-pict-lib`.

```
(relation->pict relation-name)
```

This produces a pict, but without setting `dc-for-text-size`. It is suitable for use in Slideshow or other libraries that combine `pict`s.

```
(judgment-form->pict judgment-form) → pict?
  judgment-form : judgment-form?
```

Produces a pict like `render-judgment-form`, but without setting `dc-for-text-size`. It is suitable for use in Slideshow or other libraries that combine `pict`s.

### 4.8.2 Customization

```
(render-language-nts) → (or/c #f (listof symbol?))
(render-language-nts nts) → void?
  nts : (or/c #f (listof symbol?))
```

The value of this parameter controls which non-terminals `render-language` and `language->pict` render by default. If it is `#f` (the default), all non-terminals are rendered. If it is a list of symbols, only the listed symbols are rendered.

See also `language-nts`.

```
(non-terminal-gap-space) → real?
(non-terminal-gap-space gap-space) → void?
  gap-space : real?
```

Controls the amount of vertical space between non-terminals in a typeset language.

Defaults to `0`.

Added in version 1.1 of package `redex-pict-lib`.

```
(language-make-::=-pict) → (-> (listof symbol?) pict?)
(language-make-::=-pict make-::=) → void?
  make-::= : (-> (listof symbol?) pict?)
```

Controls the pict used after the names of the non-terminals and before the first production in a grammar.

Defaults to

```
(λ (nt-names)
  ((current-text) " ::= "
                  (grammar-style)
                  (default-font-size)))
```

Added in version 1.17 of package `redex-pict-lib`.

```
(extend-language-show-union) → boolean?
(extend-language-show-union show?) → void?
  show? : boolean?
```

A parameter that controls the rendering of extended languages. If the parameter value is `#t`, then a language constructed with `define-extended-language` is shown as if the language had been constructed directly with `define-language`. If it is `#f`, then only the last extension to the language is shown (with four-period ellipses, just like in the concrete syntax).

Defaults to `#f`.

Note that the `#t` variant can look a little bit strange if `....` are used and the original version of the language has multi-line right-hand sides.

```
(extend-language-show-extended-order) → boolean?
(extend-language-show-extended-order ext-order?) → void?
  ext-order? : boolean?
```

A parameter that controls the rendering of extended languages when `extend-language-show-union` has a true value. If this parameter's value is `#t`, then productions are shown as ordered in the language extension instead of the order of the original, unextended language.

Defaults to `#f`.

Added in version 1.2 of package `redex-pict-lib`.

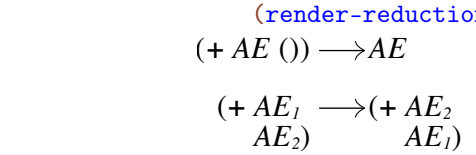
```
(render-reduction-relation-rules)
 → (or/c #f
         (listof (or/c symbol?
                       string?
                       exact-nonnegative-integer?)))
(render-reduction-relation-rules rules) → void?
  rules : (or/c #f
                (listof (or/c symbol?
                              string?
                              exact-nonnegative-integer?)))
```

This parameter controls which rules in a reduction relation will be rendered. The strings and symbols match the names of the rules and the integers match the position of the rule in the original definition.

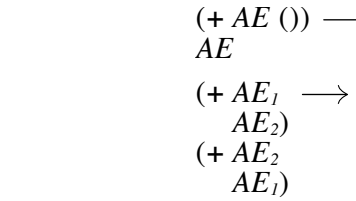
```
(rule-pict-style) → reduction-rule-style/c
(rule-pict-style style) → void?
  style : reduction-rule-style/c
```

This parameter controls the style used by default for the reduction relation. It can be `'horizontal`, where the left and right-hand sides of the reduction rule are beside each other or `'vertical`, where the left and right-hand sides of the reduction rule are above each other.

```
> (parameterize ([rule-pict-style 'horizontal])
    (render-reduction-relation simplify-ae))
```

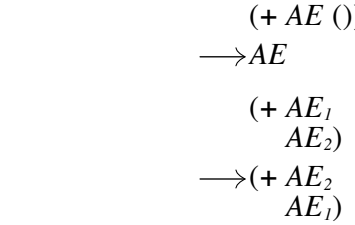


```
> (parameterize ([rule-pict-style 'vertical])
    (render-reduction-relation simplify-ae))
```



The `'compact-vertical` style moves the reduction arrow to the second line and uses less space between lines.

```
> (parameterize ([rule-pict-style 'compact-vertical])
    (render-reduction-relation simplify-ae))
```



In the `'vertical-overlapping-side-conditions` variant, the side-conditions don't contribute to the width of the pict, but are just overlaid on the second line of each rule.

The `'horizontal-left-align` style is like the `'horizontal` style, but the left-hand sides of the rules are aligned on the left, instead of on the right.

```
> (parameterize ([rule-pict-style 'horizontal-left-align])
    (render-reduction-relation simplify-ae))
```
$(+\ AE\ ()) \longrightarrow AE$

$(+\ AE_1\ AE_2) \longrightarrow (+\ AE_2\ AE_1)$

The `'horizontal-side-conditions-same-line` is like `'horizontal`, except that side-conditions are on the same lines as the rule, instead of on their own line below.

```
reduction-rule-style/c : contract?
```

A contract equivalent to

```
(or/c 'vertical
      'compact-vertical
      'vertical-overlapping-side-conditions
      'horizontal
      'horizontal-left-align
      'horizontal-side-conditions-same-line
      (-> (listof rule-pict-info?) pict-convertible?))
```

The symbols indicate various pre-defined styles. The procedure implements new styles; it is give the `rule-pict-info?` values, one for each clause in the reduction relation, and is expected to combine them into a single `pict-convertible?`

```
(rule-pict-info? x) → boolean?
  x : any/c
```

A predicate that recognizes information about a rule for use in rendering the rule as a `pict-convertible?`.

```
(rule-pict-info-arrow rule-pict-info) → symbol?
  rule-pict-info : rule-pict-info?
```

Extracts the arrow used for this rule. See also `arrow->pict`.

```
(rule-pict-info-lhs rule-pict-info) → pict-convertible?
  rule-pict-info : rule-pict-info?
```

Extracts a pict for the left-hand side of this rule.

```
(rule-pict-info-rhs rule-pict-info) → pict-convertible?
  rule-pict-info : rule-pict-info?
```

Extracts a pict for the right-hand side of this rule.

```
(rule-pict-info-label rule-pict-info) → (or/c symbol? #f)
  rule-pict-info : rule-pict-info?
```

Returns the label used for this rule, unless there is no label for the rule or *computed-label* was used, in which case this returns #f.

```
(rule-pict-info-computed-label rule-pict-info)
 → (or/c pict-convertible? #f)
  rule-pict-info : rule-pict-info?
```

Returns a pict for the typeset version of the label of this rule, when *computed-label* was used. Otherwise, returns #f.

```
(rule-pict-info->side-condition-pict  rule-pict-info
                                      [max-width])
 → pict-convertible?
  rule-pict-info : rule-pict-info?
  max-width : real? = +inf.0
```

Builds a pict for the *side-condition*s and *where* clauses for *rule-pict-info*, attempting to keep the width under *max-width*.

```
(arrow-space) → natural-number/c
(arrow-space space) → void?
  space : natural-number/c
```

This parameter controls the amount of extra horizontal space around the reduction relation arrow. Defaults to 0.

```
(label-space) → natural-number/c
(label-space space) → void?
  space : natural-number/c
```

This parameter controls the amount of extra space before the label on each rule, except in the 'vertical and 'vertical-overlapping-side-conditions modes, where it has no effect. Defaults to 0.

```
(metafunction-pict-style)
 → (or/c 'left-right
         'up-down
         'left-right/vertical-side-conditions
         'up-down/vertical-side-conditions
         'left-right/compact-side-conditions
         'up-down/compact-side-conditions
         'left-right/beside-side-conditions)
(metafunction-pict-style style) → void?
  style : (or/c 'left-right
                'up-down
                'left-right/vertical-side-conditions
                'up-down/vertical-side-conditions
                'left-right/compact-side-conditions
                'up-down/compact-side-conditions
                'left-right/beside-side-conditions)
```

This parameter controls the style used for typesetting metafunctions. The `'left-right` style means that the results of calling the metafunction are displayed to the right of the arguments and the `'up-down` style means that the results are displayed below the arguments.

```
> (parameterize ([metafunction-pict-style 'left-right])
    (render-metafunction add #:contract? #t))
```

$$
\begin{aligned}
&\mathsf{add} : K\ K \to K \\
&\mathsf{add}[\![K, \cdot]\!] = K \\
&\mathsf{add}[\![\cdot, K]\!] = K \\
&\mathsf{add}[\![(0\ K_1), (0\ K_2)]\!] = (0\ (\mathsf{add}\ K_1\ K_2)) \\
&\mathsf{add}[\![(1\ K_1), (0\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\
&\mathsf{add}[\![(0\ K_1), (1\ K_2)]\!] = (1\ (\mathsf{add}\ K_1\ K_2)) \\
&\mathsf{add}[\![(1\ K_1), (1\ K_2)]\!] = (0\ (\mathsf{add}\ (1\ \cdot)\ (\mathsf{add}\ K_1\ K_2)))
\end{aligned}
$$

```
> (parameterize ([metafunction-pict-style 'up-down])
    (render-metafunction add #:contract? #t))
```

add : $K\ K \to K$
add⟦$K$, ·⟧ =
$K$
add⟦·, $K$⟧ =
$K$
add⟦(0 $K_1$), (0 $K_2$)⟧ =
(0 (add $K_1$ $K_2$))
add⟦(1 $K_1$), (0 $K_2$)⟧ =
(1 (add $K_1$ $K_2$))
add⟦(0 $K_1$), (1 $K_2$)⟧ =
(1 (add $K_1$ $K_2$))
add⟦(1 $K_1$), (1 $K_2$)⟧ =
(0 (add (1 ·) (add $K_1$ $K_2$)))

The `'left-right/vertical-side-conditions` and `'up-down/vertical-side-conditions` variants format side conditions each on a separate line, instead of all on the same line. The `'left-right/beside-side-conditions` variant is like `'left-right`, except it puts the side-conditions on the same line, instead of on a new line below the case.

```
> (define-metafunction nums
    to-nat/sc : K -> natural
    [(to-nat/sc ·) 0]
    [(to-nat/sc (k K))
     ,(* 2 (term natural_K))
     (where k 0)
     (where natural_K (term (to-nat/sc K)))]
    [(to-nat/sc (k K))
     ,(+ 1 (* 2 (term natural_K)))
     (where k 1)
     (where natural_K (term (to-nat/sc K)))])
> (parameterize ([metafunction-pict-style 'left-right])
    (render-metafunction to-nat/sc #:contract? #t))
```

to-nat/sc : $K \to$ *natural*
to-nat/sc⟦·⟧ = 0
to-nat/sc⟦($k$ $K$)⟧ = (* 2 $natural_K$)
where $k = 0$, $natural_K$ = (term (to-nat/sc $K$))
to-nat/sc⟦($k$ $K$)⟧ = (+ 1 (* 2 $natural_K$))
where $k = 1$, $natural_K$ = (term (to-nat/sc $K$))

```
> (parameterize ([metafunction-pict-style 'left-right/vertical-
side-conditions])
    (render-metafunction to-nat/sc #:contract? #t))
```

to-nat/sc : $K \to natural$
to-nat/sc⟦·⟧ = 0
to-nat/sc⟦($k$ $K$)⟧ = (* 2 $natural_K$)
 where $k = 0$,
 $natural_K$ = (term (to-nat/sc $K$))
to-nat/sc⟦($k$ $K$)⟧ = (+ 1 (* 2 $natural_K$))
 where $k = 1$,
 $natural_K$ = (term (to-nat/sc $K$))

```
> (parameterize ([metafunction-pict-style 'left-right/beside-side-
conditions])
    (render-metafunction to-nat/sc #:contract? #t))
```

to-nat/sc : $K \to natural$
to-nat/sc⟦·⟧ = 0
to-nat/sc⟦($k$ $K$)⟧ = (* 2 $natural_K$) where $k = 0$, $natural_K$ = (term (to-nat/sc $K$))
to-nat/sc⟦($k$ $K$)⟧ = (+ 1 (* 2 $natural_K$)) where $k = 1$, $natural_K$ = (term (to-nat/sc $K$))

Sometimes, some cases have side-conditions that are wider than other cases in such a way that they should break across lines differently in different cases. The 'left-right/compact-side-conditions and 'up-down/compact-side-conditions variants move side conditions to separate lines to avoid making the rendered form wider would be otherwise—except that the rendered form is allowed to be up to the width specified by metafunction-fill-acceptable-width.

```
(metafunction-up/down-indent) → (>=/c 0)
(metafunction-up/down-indent indent) → void?
  indent : (>=/c 0)
```

Controls the indentation of the right-hand side clauses when typesetting metafunctions in one of the up/down styles (see metafunction-pict-style).

The value is the amount to indent and it defaults to 0.

Added in version 1.2 of package redex-pict-lib.

```
(delimit-ellipsis-arguments?) → any/c
(delimit-ellipsis-arguments? delimit?) → void?
  delimit? : any/c
```

This parameter controls the typesetting of metafunction definitions and applications. When it is non-#f (the default), commas precede ellipses that represent argument sequences; when it is #f no commas appear in those positions.

```
(white-square-bracket) → (-> boolean? pict-convertible?)
(white-square-bracket make-white-square-bracket) → void?
  make-white-square-bracket : (-> boolean? pict-convertible?)
```

This parameter controls the typesetting of the brackets in metafunction definitions and applications. It is called to supply the two white bracket picts. If `#t` is supplied, the function should return the open white bracket (to be used at the left-hand side of an application) and if `#f` is supplied, the function should return the close white bracket.

It's default value is `default-white-square-bracket`. See also `homemade-white-square-bracket`.

Added in version 1.1 of package `redex-pict-lib`.

```
(homemade-white-square-bracket open?) → pict-convertible?
  open? : boolean?
```

This function implements the default way that older versions of Redex typeset whitebrackets. It uses two overlapping `[` and `]` chars with a little whitespace between them.

Added in version 1.1 of package `redex-pict-lib`.

```
(default-white-square-bracket open?) → pict-convertible?
  open? : boolean?
```

This function returns picts built using `[[` and `]]` in the style `default-style`, using `current-text` and `default-font-size`.

If these result in picts that are more than 1/2 whitespace, then 1/3 of the whitespace is trimmed from sides (trimmed only from the left of the open and the right of the close).

Added in version 1.1 of package `redex-pict-lib`.

```
(linebreaks) → (or/c #f (listof boolean?))
(linebreaks breaks) → void?
  breaks : (or/c #f (listof boolean?))
```

This parameter controls which cases in the metafunction are rendered on two lines and which are rendered on one.

If its value is a list, the length of the list must match the number of cases plus one if there is a contract that is rendered. Each boolean indicates if that case has a linebreak or not.

This parameter's value influences the `'left/right` styles only.

```
(sc-linebreaks) → (or/c #f (listof boolean?))
(sc-linebreaks breaks) → void?
  breaks : (or/c #f (listof boolean?))
```

This parameter controls which cases in the metafunction have the side-conditions rendered on the next line instead of the same line as the right-hand side of the metafunction clause.

Its value must have the same shape as the value of the `linebreaks` parameter.

This parameter's value influences the `'left-right/beside-side-conditions` style only.

Added in version 1.6 of package `redex-pict-lib`.

```
(metafunction-cases)
 → (or/c #f (and/c (listof (or/c symbol?
                                 string?
                                 exact-nonnegative-integer?))
                   pair?))
(metafunction-cases cases) → void?
  cases : (or/c #f (and/c (listof (or/c symbol?
                                        string?
                                        exact-nonnegative-integer?))
                          pair?))
```

Controls which cases in a metafunction are rendered. If it is `#f` (the default), then all of the cases appear. If it is a list, then only the selected cases appear. The numbers indicate the cases counting from `0` and the strings and symbols indicate cases named with `clause-name`.

This parameter also controls how which clauses in judgment forms are rendered, but only in the case that `judgment-form-cases` is `#f` (and in that case, only the numbers are used).

```
(judgment-form-cases)
 → (or/c #f
         (non-empty-listof (or/c symbol?
                                 string?
                                 exact-nonnegative-integer?)))
(judgment-form-cases cases) → void?
  cases : (or/c #f
                (non-empty-listof (or/c symbol?
                                        string?
                                        exact-nonnegative-integer?)))
```

Controls which clauses in a judgment form are rendered. If it is `#f` (the default), then all of them are rendered. If it is a list, then only the selected clauses appear (numbers count from `0`, and strings and symbols correspond to the labels in a judgment form).

```
(judgment-form-show-rule-names) → boolean?
(judgment-form-show-rule-names show-rule-names?) → void?
  show-rule-names? : boolean?
```

Determines if the names of the cases are shown beside the rules in a rendered judgment form. Defaults to `#t`.

Added in version 1.5 of package `redex-pict-lib`.

```
(judgment-form-no-bar-when-no-premises) → boolean?
(judgment-form-no-bar-when-no-premises no-bar-when-no-premises?)
 → void?
  no-bar-when-no-premises? : boolean?
```

Controls if a horizontal bar is shown above the conclusion when there are no premises for a particular rule. If the horizontal bar is not shown, then the label for the rule is also not shown.

Used by `default-relation-clause-combine` and defaults to `#t`.

```
(label-style) → text-style/c
(label-style style) → void?
  style : text-style/c
(grammar-style) → text-style/c
(grammar-style style) → void?
  style : text-style/c
(paren-style) → text-style/c
(paren-style style) → void?
  style : text-style/c
(literal-style) → text-style/c
(literal-style style) → void?
  style : text-style/c
(metafunction-style) → text-style/c
(metafunction-style style) → void?
  style : text-style/c
(non-terminal-style) → text-style/c
(non-terminal-style style) → void?
  style : text-style/c
(non-terminal-subscript-style) → text-style/c
(non-terminal-subscript-style style) → void?
  style : text-style/c
(non-terminal-superscript-style) → text-style/c
(non-terminal-superscript-style style) → void?
  style : text-style/c
(default-style) → text-style/c
(default-style style) → void?
  style : text-style/c
```

These parameters determine the font used for various text in the picts. See `text` in the texpict collection for documentation explaining `text-style/c`. One of the more useful things a style can be is the symbol `'roman`, `'swiss`, or `'modern`, which corresponds to serif, sans-serif, and monospaced font, respectively. (A style can encode additional information, too, such as boldface or italic configuration.)

The label-style parameter is used for reduction-rule labels. The literal-style parameter is used for names that aren't non-terminals that appear in patterns. The metafunction-style parameter is used for the names of metafunctions. The paren-style parameter is used for parentheses (including "[", "]", "{", and "}", as well as "(" and ")") and for keywords, but it is not used for the square brackets of in-hole decompositions, which use the default-style parameter. The grammar-style parameter is used for the "::=" and "|" in grammars.

The non-terminal-style parameter is used for the names of non-terminals. Two parameters style the text in the (optional) "underscore" component of a non-terminal reference. The first, non-terminal-subscript-style, applies to the segment between the underscore and the first caret (^) to follow it; the second, non-terminal-superscript-style, applies to the segment following that caret. For example, in the non-terminal reference x_y^z, x has style non-terminal-style, y has style non-terminal-subscript-style, and z has style non-terminal-superscript-style. The only exception to this is when the subscript section consists only of unicode prime characters (′), in which case the non-terminal-style is used instead of the non-terminal-subscript-style.

The default-style parameter is used for parenthesis, the dot in dotted lists, spaces, the "where" and "fresh" in side-conditions, and other places where the other parameters aren't used.

Changed in version 1.4 of package redex-pict-lib: Use paren-style for keywords.

```
(label-font-size) → (and/c (between/c 1 255) integer?)
(label-font-size size) → void?
  size : (and/c (between/c 1 255) integer?)
(metafunction-font-size) → (and/c (between/c 1 255)
                                  integer?)
(metafunction-font-size size) → void?
  size : (and/c (between/c 1 255)
                integer?)
(default-font-size) → (and/c (between/c 1 255) integer?)
(default-font-size size) → void?
  size : (and/c (between/c 1 255) integer?)
```

Parameters that control the various font sizes. The default-font-size is used for all of the font sizes except labels and metafunctions.

```
(reduction-relation-rule-separation) → (parameter/c real?)
(reduction-relation-rule-separation sep) → void?
  sep : (parameter/c real?)
```

Controls the amount of space between rule in a reduction relation. Defaults to 4.

Horizontal and compact-vertical renderings add this parameter's amount to (reduction-relation-rule-extra-separation) to compute the full separation.

```
(reduction-relation-rule-extra-separation)
 → (parameter/c real?)
(reduction-relation-rule-extra-separation sep) → void?
  sep : (parameter/c real?)
```

Controls the amount of space between rule in a reduction relation for a horizontal or compact-vertical rendering, in addition to (reduction-relation-rule-separation). Defaults to 4.

Added in version 1.7 of package redex-pict-lib.

```
(reduction-relation-rule-line-separation)
 → (parameter/c real?)
(reduction-relation-rule-line-separation sep) → void?
  sep : (parameter/c real?)
```

Controls the amount of space between lines within a reduction-relation rule. Defaults to 2.

Added in version 1.7 of package redex-pict-lib.

```
(curly-quotes-for-strings) → boolean?
(curly-quotes-for-strings on?) → void?
  on? : boolean?
```

Controls if the open and close quotes for strings are turned into ‘ and ’ or are left as merely ".

Defaults to #t.

```
(current-text)
 → (-> string? text-style/c number? pict-convertible?)
(current-text proc) → void?
  proc : (-> string? text-style/c number? pict-convertible?)
```

A parameter whose value is a function to be called whenever Redex typesets some part of a grammar, reduction relation, or metafunction. It defaults to the pict library's text function.

```
(arrow->pict arrow) → pict-convertible?
  arrow : symbol?
```

Returns the pict corresponding to arrow.

```
(set-arrow-pict! arrow proc) → void?
  arrow : symbol?
  proc : (-> pict-convertible?)
```

Sets the pict for a given reduction-relation symbol. When typesetting a reduction relation that uses the symbol, the thunk will be invoked to get a pict to render it. The thunk may be invoked multiple times when rendering a single reduction relation.

```
(white-bracket-sizing)
 → (-> string? number? (values number? number? number? number?))
(white-bracket-sizing proc) → void?
  proc : (-> string? number? (values number? number? number? number?))
```

A parameter whose value is a function to be used when typesetting metafunctions to determine how to create the [[]] characters with `homemade-white-square-bracket`, which combines two `[` characters or two `]` characters together.

The procedure accepts a string that is either `"["` or `"]"`, and it returns four numbers. The first two numbers determine the offset (from the left and from the right respectively) for the second square bracket, and the second two numbers determine the extra space added (to the left and to the right respectively).

The default value of the parameter is:

```
(λ (str size)
  (let ([inset-amt (floor/even (max 4 (* size 1/2)))])
    (cond
      [(equal? str "[")
       (values inset-amt
               0
               0
               (/ inset-amt 2))]
      [else
       (values 0
               inset-amt
               (/ inset-amt 2)
               0)])))
```

where `floor/even` returns the nearest even number below its argument. This means that for sizes 9, 10, and 11, `inset-amt` will be 4, and for 12, 13, 14, and 15, `inset-amt` will be 6.

```
(horizontal-bar-spacing)
 → (parameter/c exact-nonnegative-integer?)
(horizontal-bar-spacing space) → void?
  space : (parameter/c exact-nonnegative-integer?)
```

Controls the amount of space around the horizontal bar when rendering a relation (that was created by `define-relation`). Defaults to `4`.

```
(premise-spacing) → (parameter/c exact-nonnegative-integer?)
(premise-spacing space) → void?
  space : (parameter/c exact-nonnegative-integer?)
```

Controls the amount of space between premises when render judgment forms

Added in version 1.22 of package `redex-pict-lib`.

```
(metafunction-gap-space) → real?
(metafunction-gap-space gap-space) → void?
  gap-space : real?
```

Controls the amount of vertical space between different metafunctions rendered together with `render-metafunctions`.

Defaults to 2.

Added in version 1.7 of package `redex-pict-lib`.

```
(metafunction-rule-gap-space) → real?
(metafunction-rule-gap-space gap-space) → void?
  gap-space : real?
```

Controls the amount of vertical space between different rules within a metafunction as rendered with `render-metafunction` or `render-metafunctions`.

Defaults to 2.

Added in version 1.7 of package `redex-pict-lib`.

```
(metafunction-line-gap-space) → real?
(metafunction-line-gap-space gap-space) → void?
  gap-space : real?
```

Controls the amount of vertical space between different lines within a metafunction rule as rendered with `render-metafunction` or `render-metafunctions`.

Defaults to 2.

Added in version 1.7 of package `redex-pict-lib`.

```
(metafunction-fill-acceptable-width) → real?
(metafunction-fill-acceptable-width width) → void?
  width : real?
```

Determines a width that is used for putting metafunction side conditions on a single line when using a style like `'left-right/compact-side-conditions` (as the value of `metafunction-pict-style`). The default value is `0`, which means that side conditions are joined on a line only when joining them does not change the overall width of the rendered metafunction. A larger value allows side conditions to be joined when they would make the rendered form wider, as long as the overall width of the metafunction does not exceed the specified value.

For example, if the side conditions of a particular rule in a metafunction are all shorter than the rule itself, `metafunction-fill-acceptable-width` has no effect. In contrast, if the rule itself is shorter than the side conditions and narrower than the space available to render (in a document for printing, for example), setting `metafunction-fill-acceptable-width` can help. Setting it to the available width causes rendering to use the available horizontal space for joining side conditions.

Examples:

```
> (define-metafunction nums
    [(f K_1)
     ·
     (where (0 K_2) K_1)
     (where (1 K_3) K_2)
     (where (0 K_4) K_3)
     (where (1 K_5) K_4)
     (where (1 ·) K_5)]
    [(f K) (0 ·)])
> (parameterize ([metafunction-pict-style 'left-right/compact-
side-conditions])
    (render-metafunction f))
f⟦K_1⟧ = ·
 where (0 K_2) = K_1,
       (1 K_3) = K_2,
       (0 K_4) = K_3,
       (1 K_5) = K_4,
       (1 ·) = K_5
f⟦K⟧ = (0 ·)
> (parameterize ([metafunction-pict-style 'left-right/compact-
side-conditions]
                 [metafunction-fill-acceptable-width 300])
    (render-metafunction f))
f⟦K_1⟧ = ·
 where (0 K_2) = K_1, (1 K_3) = K_2, (0 K_4) = K_3,
       (1 K_5) = K_4, (1 ·) = K_5
f⟦K⟧ = (0 ·)
> (parameterize ([metafunction-pict-style 'left-right/compact-
side-conditions]
                 [metafunction-fill-acceptable-width 400])
```

```
    (render-metafunction f))
```
$\mathsf{f}[\![K_1]\!] = \cdot$
 where $(0\ K_2) = K_1$, $(1\ K_3) = K_2$, $(0\ K_4) = K_3$, $(1\ K_5) = K_4$, $(1\ \cdot) = K_5$
$\mathsf{f}[\![K]\!] = (0\ \cdot)$

Added in version 1.11 of package `redex-pict-lib`.

```
(metafunction-combine-contract-and-rules)
 → (pict-convertible? pict-convertible? . -> . pict-convertible?)
(metafunction-combine-contract-and-rules combine) → void?
  combine : (pict-convertible? pict-convertible? . -> . pict-convertible?)
```

Controls the combination of a contract with the rules of a metafunction when contract rendering is enabled. The first argument to the combining function is a pict for the contract, and the second argument is a pict for the rules.

The default combining function uses `vl-append` with a separation of `(metafunction-rule-gap-space)`.

Added in version 1.7 of package `redex-pict-lib`.

```
(relation-clause-combine)
 → (parameter/c
    (-> (listof (listof pict-convertible?))
        pict-convertible?
        (or/c string? #f)
        pict-convertible?))
(relation-clause-combine combine) → void?
  combine : (parameter/c
             (-> (listof (listof pict-convertible?))
                 pict-convertible?
                 (or/c string? #f)
                 pict-convertible?))
```

Controls the construction of a particular clause of a reduction relation or judgment form. The first argument are the premises (each inner list of premises are on the same line as each other), the second argument is the conclusion and the third argument is the name of the rule (if the rule is named).

The default value is `default-relation-clause-combine`.

Added in version 1.9 of package `redex-pict-lib`.

```
(default-relation-clause-combine premises
                                 conclusion
                                 rule-name)
```

```
 → pict-convertible?
  premises : (listof (listof pict-convertible?))
  conclusion : pict-convertible?
  rule-name : (or/c string? #f)
```

Builds a pict for the premises as

```
(apply vc-append 4
       (for/list ([premises (in-list premises)])
         (apply hbl-append (premise-spacing) premises)))
```

and then adds a line below it and the conclusion pict below that. If *rule-name* is not `#f`, then it adds the name next to the bar.

Added in version 1.9 of package `redex-pict-lib`.

```
(relation-clauses-combine)
 → (parameter/c (-> (listof pict-convertible?) pict-convertible?))
(relation-clauses-combine combine) → void?
  combine : (parameter/c (-> (listof pict-convertible?) pict-convertible?))
```

The *combine* function is called with the list of picts that are obtained by rendering a relation; it should put them together into a single pict. It defaults to `(λ (l) (apply vc-append 20 l))`

```
(metafunction-arrow-pict)
 → (parameter/c (-> pict-convertible?))
(metafunction-arrow-pict make-arrow) → void?
  make-arrow : (parameter/c (-> pict-convertible?))
```

Specifies the pict to use for the arrow when typesetting a metafunction contract.

```
(where-make-prefix-pict)
 → (parameter/c (-> pict-convertible?))
(where-make-prefix-pict make-prefix) → void?
  make-prefix : (parameter/c (-> pict-convertible?))
```

The *make-prefix* function is called with no arguments to generate a pict that prefixes `where` clauses. It defaults to a function that produces a pict for “where” surrounded by spaces using the default style.

```
(where-combine)
 → (parameter/c (-> pict-convertible? pict-convertible? pict-convertible?))
(where-combine combine) → void?
  combine : (parameter/c (-> pict-convertible? pict-convertible? pict-convertible?))
```

The *combine* function is called with picts for the left and right side of a where clause, and it should put them together into a single pict. It defaults to (λ (l r) (hbl-append l =-pict r)), where =-pict is an equal sign surrounded by spaces using the default style.

```
(current-render-pict-adjust)
 → (pict-convertible? symbol? . -> . pict-convertible?)
(current-render-pict-adjust adjust) → void?
  adjust : (pict-convertible? symbol? . -> . pict-convertible?)
```

A parameter whose value is a function to adjusts picts generated as various parts of a rendering. The symbol that is provided to the function indicates the role of the pict. A pict-adjusting function might be installed to ensure consistent spacing among multiple lines in a metafunction's rendering, for example, or to adjust the color of side-condition terms.

The set of roles is meant to be extensible, and the currently provided role symbols are as follows:

- 'lw-line — a line with a render term (including any term that fits on a single line)
- 'language-line — a line on the right-hand side of a production in a language grammar.
- 'language-production — a production (possibly multiple lines) within a language grammar.
- 'side-condition-line — a line within a side condition for a reduction-relation rule or metafunction rule
- 'side-condition — a single side condition with a group of side conditions for a reduction-relation rule or a metafunction rule
- 'side-conditions — a group of side conditions for a reduction-relation rule or a metafunction rule including the "where" prefix added by (where-make-prefix-pict)
- 'reduction-relation-line — a single line within a reduction-relation rule
- 'reduction-relation-rule — a single rule within a reduction relation
- 'metafunction-contract — a contract for a metafunction
- 'metafunction-line — a line within a metafunction rule
- 'metafunction-rule — a single rule within a metafunction
- 'metafunctions-metafunction — a single metafunction within a group of metafunctions that are rendered together

Added in version 1.7 of package redex-pict-lib.

### 4.8.3 Removing the Pink Background

When reduction rules, a metafunction, or a grammar contains unquoted Racket code or side-conditions, they are rendered with a pink background as a guide to help find them and provide an alternative typesetting for them. In general, a good goal for a PLT Redex program that you intend to typeset is to only include such things when they correspond to standard mathematical operations, and the Racket code is an implementation of those operations.

To replace the pink code, use:

```
(with-unquote-rewriter proc expression)
```

Installs *proc* as the current unquote rewriter and evaluates *expression*. If that expression computes any picts, the unquote rewriter specified is used to remap them.

The *proc* must match the contract `(-> lw? lw?)`. Its result should be the rewritten version version of the input.

```
(with-atomic-rewriter name-symbol
                      string-or-thunk-returning-pict
                      expression)
```

Extends the current set of atomic-rewriters with one new one that rewrites the value of name-symbol to `string-or-pict-returning-thunk` (applied, in the case of a thunk), during the evaluation of expression.

*name-symbol* is expected to evaluate to a symbol. The value of *string-or-thunk-returning-pict* is used whenever the symbol appears in a pattern.

Examples:

```
> (define-language lam-lang
    (e (lambda (x) e)))
> (with-atomic-rewriter
   'lambda
   "λ"
   (render-term lam-lang (term (lambda (x) e))))
```
(term (λ (x) *e*))

```
(with-atomic-rewriters ([name-symbol string-or-thunk-returning-pict] ...)
                       expression)
```

Shorthand for nested `with-atomic-rewriter` expressions.

Added in version 1.4 of package `redex-pict-lib`.

```
(with-compound-rewriter name-symbol
                        proc
                        expression)
```

Extends the current set of compound-rewriters with one new one that rewrites the value of name-symbol via proc, during the evaluation of expression.

*name-symbol* is expected to evaluate to a symbol. The value of proc is called with a `(listof lw)`, and is expected to return a new `(listof (or/c lw? string? pict-convertible?))`, rewritten appropriately.

The list passed to the rewriter corresponds to the `lw` for the sequence that has name-symbol's value at its head.

The result list is constrained to have at most 2 adjacent non-`lw`s. That list is then transformed by adding `lw` structs for each of the non-`lw`s in the list (see the text just below the description of `lw` for a explanation of logical space):

- If there are two adjacent `lw`s, then the logical space between them is filled with whitespace.
- If there is a pair of `lw`s with just a single non-`lw` between them, a `lw` will be created (containing the non-`lw`) that uses all of the available logical space between the `lw`s.
- If there are two adjacent non-`lw`s between two `lw`s, the first non-`lw` is rendered right after the first `lw` with a logical space of zero, and the second is rendered right before the last `lw` also with a logical space of zero, and the logical space between the two `lw`s is absorbed by a new `lw` that renders using no actual space in the typeset version.

One useful way to take advantage of `with-compound-rewriters` is to return a list that begins and ends with `""` (the empty string). In that situation, any extra logical space that would have been just outside the sequence is replaced with an `lw` that does not draw anything at all.

Example:

```
> (with-compound-rewriter
    'eq
    (λ (lws)
      (define lhs (list-ref lws 2))
      (define rhs (list-ref lws 3))
      (list "" lhs " = " rhs ""))
    (render-judgment-form eq))
```

$$\frac{}{\cdot = \cdot}\ \text{[eq-·]}$$

$$\frac{K = \cdot}{(0\ K) = \cdot}\ \text{[eq-0-l]}$$

$$\frac{\cdot = K}{\cdot = (0\ K)}\ \text{[eq-0-r]}$$

$$\frac{K_1 = K_2}{(0\ K_1) = (0\ K_2)}\ \text{[eq-0]}$$

$$\frac{K_1 = K_2}{(1\ K_1) = (1\ K_2)}\ \text{[eq-1]}$$

```
(with-compound-rewriters ([name-symbol proc] ...)
                         expression)
```

Shorthand for nested `with-compound-rewriter` expressions.

### 4.8.4 LWs

```
(struct lw (e
            line
            line-span
            column
            column-span
            unq?
            metafunction?)
   #:extra-constructor-name make-lw
   #:mutable)
  e : (or/c string?
            symbol?
            pict-convertible?
            (listof (or/c (symbols 'spring) lw?)))
  line : exact-positive-integer?
  line-span : exact-positive-integer?
  column : exact-positive-integer?
  column-span : exact-positive-integer?
  unq? : boolean?
  metafunction? : boolean?
```

The lw data structure corresponds represents a pattern or a Racket expression that is to be typeset. The functions listed above construct lw structs, select fields out of them, and recognize them. The lw binding can be used with copy-struct.

The values of the unq? and metafunction? fields, respectively, indicate whether the lw represents an unquoted expression or a metafunction application. See to-lw for the meanings of the other fields.

```
(build-lw e line line-span column column-span) → lw?
  e : (or/c string?
            symbol?
            pict-convertible?
            (listof (or/c 'spring lw?)))
  line : exact-positive-integer?
  line-span : exact-positive-integer?
  column : exact-positive-integer?
  column-span : exact-positive-integer?
```

Like make-lw but specialized for constructing lws that do not represent unquoted expressions or metafunction applications.

```
(to-lw arg)
```

Turns *arg* into lw structs that contain all of the spacing information just as it would appear when being used to typeset.

Each sub-expression corresponds to its own lw, and the element indicates what kind of sub-expression it is. If the element is a list, then the lw corresponds to a parenthesized sequence, and the list contains a lw for the open paren, one lw for each component of the sequence and then a lw for the close parenthesis. In the case of a dotted list, there will also be a lw in the third-to-last position for the dot.

For example, this expression:

```
(a)
```

becomes this lw (assuming the above expression appears as the first thing in the file):

```
(build-lw (list (build-lw "(" 0 0 0 1)
                (build-lw 'a 0 0 1 1)
                (build-lw ")" 0 0 2 1))
          0 0 0 3)
```

If there is some whitespace in the sequence, like this one:

```
(a b)
```

then there is no lw that corresponds to that whitespace; instead there is a logical gap between the lws.

```
(build-lw (list (build-lw "(" 0 0 0 1)
                (build-lw 'a 0 0 1 1)
                (build-lw 'b 0 0 3 1)
                (build-lw ")" 0 0 4 1))
          0 0 0 5)
```

In general, identifiers are represented with symbols and parenthesis are represented with strings and `pict`s can be inserted to render arbitrary pictures.

The line, line-span, column, and column-span correspond to the logical spacing for the redex program, not the actual spacing that will be used when they are rendered. The logical spacing is only used when determining where to place typeset portions of the program. In the absence of any rewriters, these numbers correspond to the line and column numbers in the original program.

The line and column are absolute numbers from the beginning of the file containing the expression. The column number is not necessarily the column of the open parenthesis in a sequence – it is the leftmost column that is occupied by anything in the sequence. The line-span is the number of lines, and the column span is the number of columns on the last line (not the total width).

When there are multiple lines, lines are aligned based on the logical space (i.e., the line/column & line-span/column-span) fields of the lws. As an example, if this is the original pattern:

```
(all good boys
     deserve fudge)
```

then the leftmost edges of the words "good" and "deserve" will be lined up underneath each other, but the relative positions of "boys" and "fudge" will be determined by the natural size of the words as they rendered in the appropriate font.

When `'spring` appears in the list in the `e` field of a `lw` struct, then it absorbs all of the space around it. It is also used by `to-lw` when constructing the picts for unquoted strings. For example, this expression

```
,x
```

corresponds to these structs:

```
(build-lw (list (build-lw "" 1 0 9 0)
                'spring
                (build-lw x 1 0 10 1))
          1 0 9 2)
```

and the 'spring causes there to be no space between the empty string and the x in the typeset output.

```
(to-lw/stx stx) → lw?
  stx : syntax?
```

A procedure variant of to-lw; it accepts a syntax object and returns the corresponding lw structs. It only uses the location information in the syntax object, so metafunctions will not be rendered properly.

```
(render-lw language/nts lw) → pict-convertible?
  language/nts : (or/c (listof symbol?) compiled-lang?)
  lw : lw?
```

Produces a pict that corresponds to the *lw* object argument, using *language/nts* to determine which of the identifiers in the *lw* argument are non-terminals.

This function sets dc-for-text-size. See also lw->pict.

```
(lw->pict language/ntw lw) → pict-convertible?
  language/ntw : (or/c (listof symbol?) compiled-lang?)
  lw : lw?
```

Produces a pict that corresponds to the *lw* object argument, using language/nts to determine which of the identifiers in the *lw* argument are non-terminals.

This function does not set the dc-for-text-size parameter. See also render-lw.

```
(just-before stuff lw) → lw?
  stuff : (or/c pict-convertible? string? symbol?)
  lw : lw?
(just-after stuff lw) → lw?
  stuff : (or/c pict-convertible? string? symbol?)
  lw : lw?
```

These two helper functions build new *lw*s whose contents are the first argument, and whose line and column are based on the second argument, making the new loc wrapper be either just before or just after that argument. The line-span and column-span of the new lw is always zero.

```
(fill-between stuff lw-before lw-after) → lw?
  stuff : (or/c pict-convertible? string? symbol?)
  lw-before : lw?
  lw-after : lw?
```

Builds a new `lw` whose content is *`stuff`* and whose location information makes it occupy all of the space between *`lw-before`* and *`lw-after`*.

If *`lw-before`* and *`lw-after`* are not on the same line, `fill-between` raises an error.

### 4.8.5 Macros and Typesetting

When you have a macro that abstracts over variations in Redex programs, then typesetting is unlikely to work without some help from your macros.

To see the issue, consider this macro abstraction over a Redex grammar:

```
> (define-syntax-rule
    (def-my-lang L prim ...)
    (define-language L
      (e ::=
         (λ (x) e)
         (e e)
         prim ...
         x)
      (x ::= variable-not-otherwise-mentioned)))
> (def-my-lang L + - *)
> (render-language L)
eject: lines going backwards (current-line 2 line 1 atom
#<pict> tokens (#(struct:string-token 0 1 "*" swiss)
#(struct:pict-token 1 0 #<pict>) #(struct:string-token 0 1
"-" swiss) #(struct:pict-token 1 0 #<pict>)
#(struct:string-token 0 1 "+" swiss) #(struct:pict-token 0 0
#<pict>) #(struct:spacer-token 0 0)))
```

Redex thinks that the grammar is going "backwards" because of the way macro expansion synthesizes source locations. In particular, in the result of the macro expansion, the third production for `e` appears to come later in the file than the fourth production and this confuses Redex, making it unable to typeset this language.

One simple, not-very-general work-around is to just avoid typesetting the parts that come from the macro arguments. For example if you move the primitives into their own non-terminal and then just avoid typesetting that, Redex can cope:

```
(define-syntax-rule
  (def-my-lang/separate-prims L prim ...)
  (define-language L
    (e ::=
       (λ (x) e)
       (e e)
```

```
      prims
      x)
    (prims ::= prim ...)
    (x ::= variable-not-otherwise-mentioned)))

(def-my-lang/separate-prims L + - *)
```

```
> (render-language L #:nts '(e x))
e ::= (λ(x)e) | (ee) | prims | x
x ::= variable-not-otherwise-mentioned
```

You can also, however, exploit Racket's macro system to rewrite the source locations in a way that tells Redex where the macro-introduced parts of the language are supposed to be, and then typesetting will work normally. For example, here is one way to do this with the original language:

```
(define-syntax (def-my-lang stx)
  (syntax-case stx ()
    [(_ L a ...)
     (let ()
       (define template
         #'(define-language L
             (e (λ (x) e)
                (e e)
                HERE
                x)
             (x variable-not-otherwise-mentioned)))
       (car
        (let loop ([stx template])
          (syntax-case stx (HERE)
            [HERE
             (let loop ([as (syntax->list #'(a ...))]
                        [pos (syntax-position stx)]
                        [col (syntax-column stx)])
               (cond
                 [(null? as) '()]
                 [else
                  (define a (car as))
                  (define span
                    (string-length
                     (symbol->string (syntax-e a))))
                  (define srcloc
                    (vector (syntax-source stx)
                            (syntax-line stx)
```

```
                    col
                    pos
                    span))
              (cons
               (datum->syntax a
                              (syntax-e a)
                              srcloc
                              a)
               (loop (cdr as)
                     (+ pos span 1)
                     (+ col span 1)))]))]
        [(a ...)
         (list
          (datum->syntax
           stx
           (apply append (map loop (syntax->list #'(a ...))))
           stx
           stx))]
        [a
         (list stx)]))))]))
```

```
> (def-my-lang L + - *)
```

```
> (render-language L)
```

$e ::= (\lambda(x)e) \mid (ee) \mid + \mid - \mid * \mid x$
$x ::=$ *variable-not-otherwise-mentioned*

# 5 Automated Testing Benchmark

```
(require redex/benchmark)
```
package: redex-benchmark

Redex's automated testing benchmark provides a collection of buggy models and falsifiable properties to test how efficiently methods of automatic test case generation are able to find counterexamples for the bugs.

Each entry in the benchmark contains a *check* function and multiple *generate* functions. The check function determines if a given example is a counterexample (i.e. if it uncovers the buggy behavior) and each of the generate functions generates candidate examples to be tried. There are multiple ways to generate terms for each model. They typically correspond to different uses of `generate-term`, but could be any way to generate examples. See `run-gen-and-check` for the precise contracts for generate and check functions.

Most of the entries in the benchmark are small differences to existing, bug-free models, where some small change to the model introduces the bug. These changes are described using `define-rewrite`.

To run a benchmark entry with a particular generator, see `run-gen-and-check/mods`.

## 5.1 The Benchmark Models

The programs in our benchmark come from two sources: synthetic examples based on our experience with Redex over the years and from models that we and others have developed and bugs that were encountered during the development process.

The benchmark has six different Redex models, each of which provides a grammar of terms for the model and a soundness property that is universally quantified over those terms. Most of the models are of programming languages and most of the soundness properties are type-soundness, but we also include red-black trees with the property that insertion preserves the red-black invariant, as well as one richer property for one of the programming language models (discussed in §5.1.3 "stlc-sub").

For each model, we have manually introduced bugs into a number of copies of the model, such that each copy is identical to the correct one, except for a single bug. The bugs always manifest as a term that falsifies the soundness property.

The table in figure 1 gives an overview of the benchmark suite, showing some numbers for each model and bug. Each model has its name and the number of lines of code for the bug-free model (the buggy versions are always within a few lines of the originals). The line number counts include the model and the specification of the property.

Each bug has a number and, with the exception of the rvm model, the numbers count from 1 up to the number of bugs. The rvm model bugs are all from Klein et al. (2013)'s work and

we follow their numbering scheme (see §5.1.8 “rvm” for more information about how we chose the bugs from that paper).

The **S/M/D/U** column shows a classification of each bug as:

- **S** (Shallow) Errors in the encoding of the system into Redex, due to typos or a misunderstanding of subtleties of Redex.
- **M** (Medium) Errors in the algorithm behind the system, such as using too simple of a data-structure that doesn't allow some important distinction, or misunderstanding that some rule should have a side-condition that limits its applicability.
- **D** (Deep) Errors in the developer's understanding of the system, such as when a type system really isn't sound and the author doesn't realize it.
- **U** (Unnatural) Errors that are unlikely to have come up in real Redex programs but are included for our own curiosity. There are only two bugs in this category.

The size column shows the size of the term representing the smallest counterexample we know for each bug, where we measure size as the number of pairs of parentheses and atoms in the s-expression representation of the term.

Each subsection of this section introduces one of the models in the benchmark, along with the errors we introduced into each model.

### 5.1.1 stlc

A simply-typed $\lambda$-calculus with base types of numbers and lists of numbers, including the constants `+`, which operates on numbers, and `cons`, `head`, `tail`, and `nil` (the empty list), all of which operate only on lists of numbers. The property checked is type soundness: the combination of preservation (if a term has a type and takes a step, then the resulting term has the same type) and progress (that well-typed non-values always take a reduction step).

We introduced nine different bugs into this system. The first confuses the range and domain types of the function in the application rule, and has the small counterexample: `(hd 0)`. We consider this to be a shallow bug, since it is essentially a typo and it is hard to imagine anyone with any knowledge of type systems making this conceptual mistake. Bug 2 neglects to specify that a fully applied `cons` is a value, thus the list `((cons 0) nil)` violates the progress property. We consider this be be a medium bug, as it is not a typo, but an oversight in the design of a system that is otherwise correct in its approach.

We consider the next three bugs to be shallow. Bug 3 reverses the range and the domain of function types in the type judgment for applications. This was one of the easiest bug for all of our approaches to find. Bug 4 assigns `cons` a result type of `int`. The fifth bug

| Model | LoC | Bug# | S/M/D/U | Size | Description of Bug |
|---|---|---|---|---|---|
| stlc | 211 | 1 | S | 3 | app rule the range of the function is matched to the argument |
| | | 2 | M | 5 | the ((cons v) v) value has been omitted |
| | | 3 | S | 8 | the order of the types in the function position of application has been swapped |
| | | 4 | S | 9 | the type of cons is incorrect |
| | | 5 | S | 7 | the tail reduction returns the wrong value |
| | | 6 | M | 7 | hd reduction acts on partially applied cons |
| | | 7 | M | 9 | evaluation isn't allowed on the rhs of applications |
| | | 8 | U | 12 | lookup always returns int |
| | | 9 | S | 15 | variables aren't required to match in lookup |
| poly-stlc | 280 | 1 | S | 6 | app rule the range of the function is matched to the argument |
| | | 2 | M | 11 | the (([cons @ $\tau$] v) v) value has been omitted |
| | | 3 | S | 14 | the order of the types in the function position of application has been swapped |
| | | 4 | S | 15 | the type of cons is incorrect |
| | | 5 | S | 16 | the tail reduction returns the wrong value |
| | | 6 | M | 16 | hd reduction acts on partially applied cons |
| | | 7 | M | 9 | evaluation isn't allowed on the rhs of applications |
| | | 8 | U | 15 | lookup always returns int |
| | | 9 | S | 18 | variables aren't required to match in lookup |
| stlc-sub | 241 | 1 | S | 8 | forgot the variable case |
| | | 2 | S | 13 | wrong order of arguments to replace call |
| | | 3 | S | 10 | swaps function and argument position in application |
| | | 4 | D | 22 | variable not fresh enough |
| | | 5 | SM | 17 | replace all variables |
| | | 6 | S | 8 | forgot the variable case |
| | | 7 | S | 13 | wrong order of arguments to replace call |
| | | 8 | S | 10 | swaps function and argument position in application |
| | | 9 | SM | 17 | replace all variables |
| let-poly | 662 | 1 | S | 8 | use a lambda-bound variable where a type variable should have been |
| | | 2 | D | 28 | the classic polymorphic let + references bug |
| | | 3 | M | 3 | mix up types in the function case |
| | | 4 | S | 8 | dropped the occurs check |
| | | 5 | M | 3 | eliminate-G was written as if it always gets a Gx as input |
| | | 6 | M | 8 | copy and paste error in the orient $\rightarrow$ rule |
| | | 7 | D | 12 | used let --> left-left-$\lambda$ rewrite rule for let, but the right-hand side is less polymo |
| list-machine | 256 | 1 | S | 22 | confuses the lhs value for the rhs value in cons type rule |
| | | 2 | M | 22 | var-set may skip a var with matching id (in reduction) |
| | | 3 | S | 29 | cons doesn't actually update the store |
| rbtrees | 187 | 1 | M | 13 | ins does no rebalancing |
| | | 2 | M | 15 | the first case is removed from balance |
| | | 3 | S | 51 | doesn't increment black depth in non-empty case |
| delim-cont | 287 | 1 | M | 46 | guarded mark reduction doesn't wrap results with a list/c |
| | | 2 | M | 25 | list/c contracts aren't applied properly in the cons case |
| | | 3 | S | 52 | the function argument to call/comp has the wrong type |
| rvm | 712 | 2 | M | 24 | stack offset / pointer confusion |
| | | 3 | D | 33 | application slots not initialized properly |
| | | 4 | M | 17 | mishandling branches when then branch needs more stack than else branch; bug |
| | | 5 | M | 23 | mishandling branches when then branch needs more stack than else branch; bug |
| | | 6 | M | 15 | forgot to implement the case-lam branch in verifier |
| | | 14 | M | 27 | certain updates to initialized slots could break optimizer assumptions |
| | | 15 | S | 21 | neglected to restrict case-lam to accept only 'val' arguments |



Figure 1: Benchmark Overview

returns the head of a list when `tl` is applied. Bug 6 only applies the `hd` constant to a partially constructed list (i.e., the term `(cons 0)` instead of `((cons 0) nil)`). Only the grammar based random generation exposed bugs 5 and 6 and none of our approaches exposed bug 4.

The seventh bug, also classified as medium, omits a production from the definition of evaluation contexts and thus doesn't reduce the right-hand-side of function applications.

Bug 8 always returns the type `int` when looking up a variable's type in the context. This bug (and the identical one in the next system) are the only bugs we classify as unnatural. We included it because it requires a program to have a variable with a type that is more complex that just `int` and to actually use that variable somehow.

Bug 9 is simple; the variable lookup function has an error where it doesn't actually compare its input to variable in the environment, so it effectively means that each variable has the type of the nearest enclosing lambda expression.

### 5.1.2 poly-stlc

This is a polymorphic version of §5.1.1 "stlc", with a single numeric base type, polymorphic lists, and polymorphic versions of the list constants. No changes were made to the model except those necessary to make the list operations polymorphic. There is no type inference in the model, so all polymorphic terms are required to be instantiated with the correct types in order for the function to type check. Of course, this makes it much more difficult to automatically generate well-typed terms, and thus counterexamples. As with **stlc**, the property checked is type soundness.

All of the bugs in this system are identical to those in **stlc**, aside from any changes that had to be made to translate them to this model.

This model is also a subset of the language specified in Pałka et al. (2011), who used a specialized and optimized QuickCheck generator for a similar type system to find bugs in GHC. We adapted this system (and its restriction in **stlc**) because it has already been used successfully with random testing, which makes it a reasonable target for an automated testing benchmark.

### 5.1.3 stlc-sub

The same language and type system as §5.1.1 "stlc", except that in this case all of the errors are in the substitution function.

Our own experience has been that it is easy to make subtle errors when writing substitution functions, so we added this set of tests specifically to target them with the benchmark. There are two soundness checks for this system. Bugs 1-5 are checked in the following way: given a candidate counterexample, if it type checks, then all $\beta$v-redexes in the term are reduced

(but not any new ones that might appear) using the buggy substitution function to get a second term. Then, these two terms are checked to see if they both still type check and have the same type and that the result of passing both to the evaluator is the same.

Bugs 4-9 are checked using type soundness for this system as specified in the discussion of the §5.1.1 "stlc" model. We included two predicates for this system because we believe the first to be a good test for a substitution function but not something that a typical Redex user would write, while the second is something one would see in most Redex models but is less effective at catching bugs in the substitution function.

The first substitution bug we introduced simply omits the case that replaces the correct variable with the term to be substituted. We considered this to be a shallow error, and indeed all approaches were able to uncover it, although the time it took to do so varied.

Bug 2 permutes the order of arguments when making a recursive call. This is also categorized as a shallow bug, although it is a common one, at least based on our experience writing substitutions in Redex.

Bug 3 swaps the function and argument positions of an application while recurring, again essentially a typo and a shallow error, although one of the more difficult to find in this model.

The fourth substitution bug neglects to make the renamed bound variable fresh enough when recurring past a lambda. Specifically, it ensures that the new variable is not one that appears in the body of the function, but it fails to make sure that the variable is different from the bound variable or the substituted variable. We categorized this error as deep because it corresponds to a misunderstanding of how to generate fresh variables, a central concern of the substitution function.

Bug 5 carries out the substitution for all variables in the term, not just the given variable. We categorized it as SM, since it is essentially a missing side condition, although a fairly egregious one.

Bugs 6-9 are duplicates of bugs 1-3 and bug 5, except that they are tested with type soundness instead. (It is impossible to detect bug 4 with this property.)

### 5.1.4 let-poly

A language with ML-style `let` polymorphism, included in the benchmark to explore the difficulty of finding the classic let+references unsoundness. With the exception of the classic bug, all of the bugs were errors made during the development of this model (and that were caught during development).

The first bug is simple; it corresponds to a typo, swapping an `x` for a `y` in a rule such that a type variable is used as a program variable.

Bug number 2 is the classic let+references bug. It changes the rule for `let`-bound variables

in such a way that generalization is allowed even when the initial value expression is not a value.

Bug number 3 is an error in the function application case where the wrong types are used for the function position (swapping two types in the rule).

Bugs 4, 5, and 6 were errors in the definition of the unification function that led to various bad behaviors.

Finally, bug 7 is a bug that was introduced early on, but was only caught late in the development process of the model. It used a rewriting rule for `let` expressions that simply reduced them to the corresponding $((\lambda$ expressions. This has the correct semantics for evaluation, but the statement of type-soundness does not work with this rewriting rule because the let expression has more polymorphism that the corresponding application expression.

### 5.1.5 list-machine

An implementation of Appel et al. (2012)'s list-machine benchmark. This is a reduction semantics (as a pointer machine operating over an instruction pointer and a store) and a type system for a seven-instruction first-order assembly language that manipulates `cons` and `nil` values. The property checked is type soundness as specified in Appel et al. (2012), namely that well-typed programs always step or halt. Three mutations are included.

The first list-machine bug incorrectly uses the head position of a cons pair where it should use the tail position in the cons typing rule. This bug amounts to a typo and is classified as simple.

The second bug is a missing side-condition in the rule that updates the store that has the effect of updating the first position in the store instead of the proper position in the store for all of the store update operations. We classify this as a medium bug.

The final list-machine bug is a missing subscript in one rule that has the effect that the list cons operator does not store its result. We classify this as a simple bug.

### 5.1.6 rbtrees

A model that implements the red-black tree insertion function and checks that insertion preserves the red-black tree invariant (and that the red-black tree is a binary search tree).

The first bug simply removes the re-balancing operation from insert. We classified this bug as medium since it seems like the kind of mistake that a developer might make in staging the implementation. That is, the re-balancing operation is separate and so might be put off initially, but then forgotten.

The second bug misses one situation in the re-balancing operation, namely when a black node has two red nodes under it, with the second red node to the right of the first. This is a medium bug.

The third bug is in the function that counts the black depth in the red-black tree predicate. It forgets to increment the count in one situation. This is a simple bug.

### 5.1.7 delim-cont

Takikawa et al. (2013)'s model of a contract and type system for delimited control. The language is Plotkin's PCF extended with operators for delimited continuations, continuation marks, and contracts for those operations. The property checked is type soundness. We added three bugs to this model.

The first was a bug we found by mining the model's git repository's history. This bug fails to put a list contract around the result of extracting the marks from a continuation, which has the effect of checking the contract that is supposed to be on the elements of a list against the list itself instead. We classify this as a medium bug.

The second bug was in the rule for handling list contracts. When checking a contract against a cons pair, the rule didn't specify that it should apply only when the contract is actually a list contract, meaning that the cons rule would be used even on non-list contacts, leading to strange contract checking. We consider this a medium bug because the bug manifests itself as a missing `list/c` in the rule.

The last bug in this model makes a mistake in the typing rule for the continuation operator. The mistake is to leave off one-level of arrows, something that is easy to do with so many nested arrow types, as continuations tend to have. We classify this as a simple error.

### 5.1.8 rvm

A existing model and test framework for the Racket virtual machine and bytecode verifier (Klein et al. 2013). The bugs were discovered during the development of the model and reported in section 7 of that paper. Unlike the rest of the models, we do not number the bugs for this model sequentially but instead use the numbers from Klein et al. (2013)'s work.

We included only some of the bugs, excluding bugs for two reasons:

- The paper tests two properties: an internal soundness property that relates the verifier to the virtual machine model, and an external property that relates the verifier model to the verifier implementation. We did not include any that require the latter properties because it requires building a complete, buggy version of the Racket runtime system to include in the benchmark.

- We included all of the internal properties except those numbered 1 and 7 for practical reasons. The first is the only bug in the machine model, as opposed to just the verifier, which would have required us to include the entire VM model in the benchmark. The second would have required modifying the abstract representation of the stack in the verifier model in contorted way to mimic a more C-like implementation of a global, imperative stack. This bug was originally in the C implementation of the verifier (not the Redex model) and to replicate it in the Redex-based verifier model would require us to program in a low-level imperative way in the Redex model, something not easily done.

These bugs are described in detail in Klein et al. (2013)'s paper.

This model is unique in our benchmark suite because it includes a function that makes terms more likely to be useful test cases. In more detail, the machine model does not have variables, but instead is stack-based; bytecode expressions also contain internal pointers that must be valid. Generating a random (or in-order) term is relatively unlikely to produce one that satisfies these constraints. For example, of the first 10,000 terms produced by the in-order enumeration only 1625 satisfy the constraints. The ad hoc random generator generators produces about 900 good terms in 10,000 attempts and the uniform random generator produces about 600 in 10,000 attempts.

To make terms more likely to be good test cases, this model includes a function that looks for out-of-bounds stack offsets and bogus internal pointers and replaces them with random good values. This function is applied to each of the generated terms before using them to test the model.

## 5.2 Managing Benchmark Modules

This section describes utilities for making changes to existing modules to create new ones, intended to assist in adding bugs to models and keeping buggy models in sync with changes to the original model.

```
(define-rewrite id from ==> to
  [#:context (context-id ...)
   #:variables (variable-id ...)
   #:once-only
   #:exactly-once])
```

Defines a syntax transformer bound to `id`, the effect of which is to rewrite syntax matching the pattern `from` to the result expression `to`. The `from` argument should follow the grammar of a `syntax-case` pattern, and `to` acts as the corresponding result expression. The behavior of the match is the same as `syntax-case`, except that all identifiers in `from` are treated as literals with the exception of an identifier that has the same binding as a `variable-id` appearing in the `#:variables` keyword argument, which is treated as a pattern variable.

(The reverse of the situation for `syntax-case`, where literals must be specified instead.) The rewrite will only be applied in the context of a `module` form, but it will be applied wherever possible within the module body, subject to a few constraints.

The rest of the keyword arguments control where and how often the rewrite may be applied. The `#:once-only` option specifies that the rewrite can be applied no more than once, and the `#:exactly-once` option asserts that the rewrite must be applied once (and no more). In both cases a syntax error is raised if the condition is not met. The `#:context` option searches for syntax of the form `(some-id . rest)`, where the binding of `some-id` matches that of the first `context-id` in the `#:context` list, at which point it recurs on `rest` but drops the first id from the list. Once every `context-id` has been matched, the rewrite can be applied.

```
(define-rewrite/compose id rw-id ...)
```

Defines a syntax transformer bound to `id`, assuming that every `rw-id` also binds a syntax transformer, such that `id` has the effect of applying all of the `rw-id`s.

```
(include/rewrite path-spec mod-id rw-id ...)
```

If the syntax designated by `path-spec` is a module, the module syntax is inlined as a submodule with the identifier `mod-id`. Assumes each `rw-id` binds a syntax transformer, and applies them to the resulting module syntax. The syntax of `path-spec` must be same as for `include`.

For example, if the contents of the file mod-fx.rkt are:

"mod-fx.rkt"

```
#lang racket/base

(provide f)
(define x 'X!)
(define (f x) x)
```

Then:

```
> (define-rewrite xy-rw
    x ==> y
    #:context (f)
    #:once-only)
> (require "mod-fx.rkt")
> (f 3)
3
> (include/rewrite "mod-fx.rkt" submod-fx xy-rw)
> (require (prefix-in s: 'submod-fx))
> (s:f 3)
'X!
```

## 5.3 Running Benchmark Models

```
(run-gen-and-check  get-gen
                    check
                    seconds
                   [#:name name
                    #:type type]) → run-results?
  get-gen : (-> (-> any/c))
  check : (-> any/c boolean?)
  seconds : natural-number/c
  name : string? = "unknown"
  type : symbol? = 'unknown
```

Repeatedly generates random terms and checks if they are counterexamples to some property defined by *check*, where a term is considered a counterexample if *check* returns #f for that term.

The *get-gen* thunk is called to build a generator of random terms (which may close over some state). A new generator is created each time the property is found to be false.

Each generated term is passed to *check* to see if it is a counterexample. The interval in milliseconds between counterexamples is tracked, and the process is repeated either until the time specified by *seconds* has elapsed or the standard error in the average interval between counterexamples is less than 10% of the average.

The result is an instance of run-results containing the total number of terms generated, the total elapsed time, and the number of counterexamples found. More detailed information can be obtained using the benchmark logging facilities, for which *name* is refers to the name of the model, and *type* is a symbol indicating the generation type used.

```
(struct run-results (tries time cexps))
  tries : natural-number/c
  time : natural-number/c
  cexps : natural-number/c
```

Minimal results for one run of a generate and check pair.

```
(run-gen-and-check/mods  gen-mod-path
                         check-mod-path
                         seconds
                        [#:name name])   → run-results?
  gen-mod-path : module-path?
  check-mod-path : module-path?
  seconds : natural-number/c
  name : string? = "unknown"
```

Just like run-gen-and-check, except that *gen-mod-path* and *check-mod-path* are module paths to a *generator module* and a *check module*, which are assumed to have the following characteristics:

- A *generator module* provides the function get-generator, which meets the specification for the get-gen argument to run-gen-and-check, and type, which is a symbol designating the type of the generator.
- A *check module* provides the function check, which meets the specification for the check argument to run-gen-and-check.

## 5.4 Logging

```
(struct bmark-log-data (data))
  data : any/c
```

Contains data logged by the benchmark, as described below.

Detailed information gathered during a benchmark run is logged to the current-logger, at the 'info level, with the message "BENCHMARK-LOGGING". The data field of the log message contains a bmark-log-data struct, which wraps data of the form:

```
log-data = (list event timestamp data-list)
```

Where event is a symbol that designates the type of event, and timestamp is symbol that contains the current-date of the event in ISO-8601 format. The information in data-list depends on the event, but must be in the form of a list alternating between a keyword and a datum, where the keyword is a short description of the datum.

The following events are logged (the symbol designating the event is in parentheses, and the form of the data logged for each event is shown):

- Run starts ('start), logged when beginning a run with a new generate/check pair.
  ```
  data-list = (list '#:model model '#:type gen)
  ```
- Run completions ('finished), logged at the end of a run.
  ```
  data-list = (list '#:model model '#:type gen '#:time-ms time
                    '#:attempts tries
                    '#:num-counterexamples countxmps
                    '#:rate-terms/s rate '#:attempts/cexp atts)
  ```
- Every counterexample found ('counterexample).

```
data-list = (list '#:model model '#:type gen
                  '#:counterexample term '#:iterations tries
                  '#:time time)
```

- New average intervals between counterexamples (`'new-average`), which are recalculated whenever a counterexample is found.

  ```
  data-list = (list '#:model model '#:type gen
                    '#:average avg '#:stderr err)
  ```

- Major garbage collections (`'gc-major`).

  ```
  data-list = (list '#:amount amount '#:time time)
  ```

- Heartbeats (`'hearbeat`) are logged every 10 seconds by the benchmark as a way to be sure that the benchmark has not crashed.

  ```
  data-list = (list '#:model model '#:type gen)
  ```

- Timeouts (`'timeout`), which occur when generating or checking a single takes term longer than 5 minutes.

  ```
  data-list = (list '#:during 'check '#:term term '#:model model
                    '#:type gen)
            | (list '#:during 'generation '#:model model '#:type gen)
  ```

```
(benchmark-logging-to filename thunk) → any/c
  filename : string?
  thunk : (-> any/c)
```

Intercepts events logged by the benchmark and writes the data specified by the `log-data` production above to *`filename`*.

```
(bmark-log-directory)
 → (or/c path-string? path-for-some-system? 'up 'same)
(bmark-log-directory directory) → void?
  directory : (or/c path-string? path-for-some-system? 'up 'same)
 = (current-directory)
```

Controls the directory where `filename` in `benchmark-logging-to` is located.

## 5.5 Plotting

Plotting and analysis tools consume data of the form produced by the benchmark logging facilities (see §5.4 “Logging”).

*TODO!*

## 5.6 Finding the Benchmark Models

```
(require redex/benchmark/models/all-info)
```
package: redex-benchmark

```
(all-mods)
 → (listof (list/c string? module-path? module-path?))
```

Returns a list of generate and check pairs for a given model or set of models, such that for each pair the first element is the name of the model, the second is a module defining a generator, and the third is a module defining a check function.

The models included in the distribution of the benchmark are in the "redex/benchmark/models" subdirectory of the redex-benchmark package. In addition to the redex/benchmark/models/all-info library documented here, each such subdirectory contains an info file named according to the pattern "<name>-info.rkt", defining a module that provides a model-specific all-mods function.

A command line interface is provided by the file "redex/benchmark/run-benchmark.rkt", which takes an "info" file as described above as its primary argument and provides options for running the listed tests. It automatically writes results from each run to a separate log file, all of which are located in a temporary directory. (The directory path is printed to standard out at the beginning of the run).

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Redex Pattern, number
Redex Pattern, natural
Redex Pattern, name
Redex Pattern, integer
Redex Pattern, in-hole
Redex Pattern, hole
Redex Pattern, hide-hole
Redex Pattern, cross
Redex Pattern, compatible-closure-context
Redex Pattern, boolean
Redex Pattern, any
Redex Pattern, _
redex-check
redex-define
redex-enum
redex-generator
redex-index
redex-let
redex-let*
redex-match
redex-match?
redex-pseudo-random-generator
redex/benchmark
redex/benchmark/models/all-info
redex/gui
redex/pict
redex/reduction-semantics
Redex: Practical Semantics Engineering
Reduction Relations
Reduction Relations
reduction-relation
reduction-relation->pict
reduction-relation->rule-names
reduction-relation-rule-extra-separation
reduction-relation-rule-line-separation
reduction-relation-rule-separation
reduction-relation?
reduction-rule-style/c
reduction-steps-cutoff
Reductions and Semantics
relation->pict
relation-clause-combine
relation-clauses-combine
relation-coverage
Removing the Pink Background
render-judgment-form
render-language
render-language-nts
render-lw
render-metafunction
render-metafunctions
render-reduction-relation
render-reduction-relation-rules
render-relation
render-term
render-term/pretty-write
rule-pict-info->side-condition-pict
rule-pict-info-arrow
rule-pict-info-computed-label
rule-pict-info-label
rule-pict-info-lhs
rule-pict-info-rhs

rule-pict-info?
rule-pict-style
run-gen-and-check
run-gen-and-check/mods
run-results (struct)
run-results-cexps
run-results-time
run-results-tries
run-results?
Running Benchmark Models
rvm
sc-linebreaks
*scope*
Semantics
*semantics*
set-arrow-pict!
set-cache-size!
set-lw-column!
set-lw-column-span!
set-lw-e!
set-lw-line!
set-lw-line-span!
set-lw-metafunction?!
set-lw-unq?!
shadow
show-derivations
side-condition
side-condition clause
side-condition/hidden clause
Solution: Binary Addition
Solution: Contracts
Solution: Finite State Machines
Solution: GC
Solution: Missionaries and Cannibals
Solution: Objects
Solution: Threads
Solution: Towers of Hanoi
Solution: Types
*Standard reduction*
stepper
stepper/seed
stlc
stlc-sub
string
struct:bind
struct:bmark-log-data
struct:counterexample
struct:derivation
struct:exn:fail:redex:test
struct:lw
struct:run-results
Subjection Reduction
substitute
Substitution
*symbol*
Syntax and Metafunctions
term
*term*
term->pict
term->pict/pretty-write
term-define
term-let
term-match
term-match/single
term-node-children
term-node-color
term-node-expr
term-node-height
term-node-labels
term-node-parents
term-node-set-color!
term-node-set-position!
term-node-set-red!
term-node-width
term-node-x
term-node-y
term-node?
Terms
test-->
test-->>
test-->>E
test-->>∃
test-equal
test-judgment-holds

test-match
test-no-match
test-predicate
test-results
Testing
Testing Reduction Relations
Testing Typing
The Benchmark Models
The CC-CK Theorem
The CEK machine
The CEK-CK Theorem
The CK Machine
The Redex Reference
The Theoretical Framework
to-lw
to-lw/stx
traces
traces/ps
*Truth*
Types
Types and Property Testing
Typesetting
Typesetting the Reduction Relation
Typing
union-reduction-relations
*Value*
variable
Variable Assignment
variable-except
variable-not-in
variable-not-otherwise-mentioned
variable-prefix
variables-not-in
What are Models
*where clause*
where-combine
where-make-prefix-pict
where/error clause
where/hidden clause
white-bracket-sizing
white-square-bracket
with
with-atomic-rewriter
with-atomic-rewriters
with-compound-rewriter
with-compound-rewriters
with-unquote-rewriter